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January 2017 | ₹ 35.00

MATHEMATICS Spectrum

TARGET 2017

Rapid
Concept Revision

Sequences & Series
Mathematical Induction
Permutations & Combinations
Binomial Theorem
Statistics

FROM CLASS 11 SYLLABUS

Higher Order Differentiation
Mean Value Theorem
Application of Derivatives
Tangent & Normal
Increasing & Decreasing Functions
Maxima & Minima

FROM CLASS 12 SYLLABUS

TEST RIDER FOR JEE
TEST DRIVE FOR BOARD

JUVENILEZ

GOLDEN OLDIES *Integration*
RAINBOW *Differential Equation*
FUN ARCADE

MEGA ISSUE 2
More Pages
More Features
More Practice...

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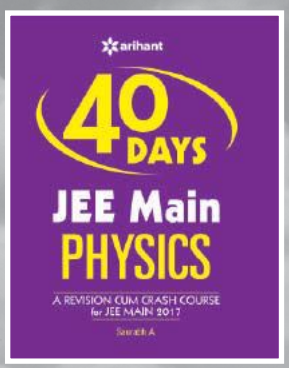
40 DAYS
JEE MAIN 2016

A Revision Cum Crash Course for JEE MAIN 2017

PHYSICS, CHEMISTRY, MATHEMATICS



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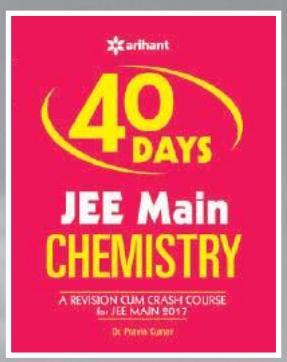

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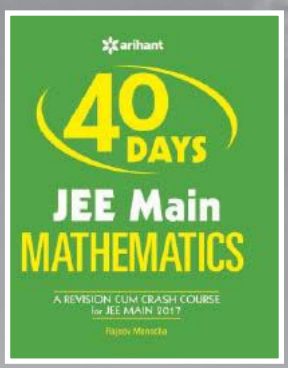

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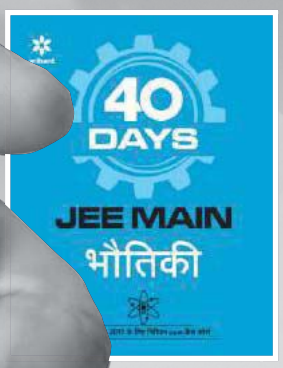

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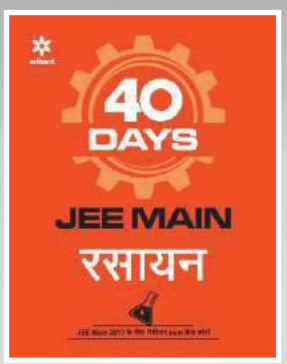

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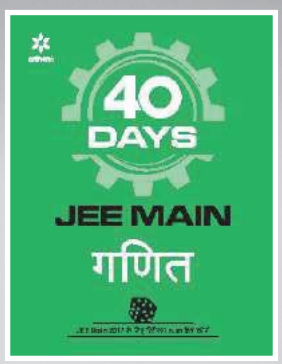

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Issue 2 **Vol 3**

Dear Aspirants,
A New Year always brings new hopes, new plans, new strategies and new resolutions to our life. I like January, because it is the month in which, I can see a whole year stretching out ahead of me. Slowly, 2016 is winding down and 2017 is knocking at our doors.

At the heart of calculus lies INFINITY signifying that strange region of Mathematics, which is part mystical, part magical; that's dangerous by reputation and ultimately romantic. The non-mathematical world uses "infinity" to stand for a confusing mass of ideas. It is because of this confusion, philosophers, scientists and priests had quarreled over infinity for thousands of years and even sometimes violently. The present world has become so competitive, you will have to do your best to secure your share of success.

From now all of you must be in the phase of revision. This last lap of 90 days is very crucial for any aspirant and from the side of spectrum family, I personally assure you that we will try to create more and more to help you out. Along with all these, one personal piece of advice from my side is

"Try to manage your Time as you manage your Money."

The spectrum team will be anxiously waiting for your feedbacks and comments.
Good luck

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HAPPY NEW YEAR 2017

@ CLASS XI SYLLABUS

Rapid CONCEPT REVISION

TARGET 2017

SEQUENCES AND SERIES

A sequence is a function, whose domain is the set of natural numbers. If the range of sequence is a subset of real numbers (complex numbers), then it is called real sequence (complex sequence). If $f : N \to C$ is a sequence, then we usually denote it as $\{f(1), f(2), \ldots, f(n)\}$. If $t_n = f(n)$, then the sequence is written as $(t_1, t_2, \ldots, t_n)$.

If $t_1, t_2, t_3, \ldots, t_n$ is a sequence, then the expression $t_1 + t_2 + t_3 + \ldots + t_n$ is a series.

Progression

A sequence whose terms follow certain rule is called progression.

Arithmetic Progression (AP)

A sequence is said to be arithmetic progression when the difference of two consecutive terms remains constant. This constant is called **common difference** of an AP and usually denoted by 'd'.

$$d = T_n - T_{n-1}, \; n \geq 2$$

The standard form of AP is $a, a + d, a + 2d, \ldots, a + (n-1)d$.

*n*th Term of an AP

If a is the first term and d is the common difference of an AP, then its nth term is given by $t_n = a + (n-1)d$.

Note (i) If l is the last term of AP, then nth term from the last is $t'_n = l - (n-1)d$.

(ii) If nth term of a sequence is a linear expression in n i.e. $T_n = an + b$, then the sequence is an AP.

Sum of *n* Terms of an AP

The sum of first n terms of an AP is given by

$$S_n = \frac{n}{2}[2a + (n-1)d] \quad \text{or} \quad S_n = \frac{n}{2}(a + t_n)$$

where, a = first term, d = common difference

n = number of terms, t_n = nth term or last term of AP.

Note If the sum of first n term of a sequence is a quadratic expression in n, then the sequence is an AP.

Properties of AP

(i) If fixed number is added to (or subtracted) from each term of a given AP, then resulting sequence is also an AP and it has same common difference as that of given AP.

(ii) If each term of an AP is multiplied (or divided) by fixed non-zero constant, then resulting sequence remains in AP with common difference constant times the common difference of original AP.

(iii) If $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$ are two AP's, then

(a) $a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots, a_n + b_n$ are also in AP.

(b) $a_1 - b_1, a_2 - b_2, a_3 - b_3, \ldots, a_n - b_n$ are also in AP.

(c) $a_1 b_1, a_2 b_2, \ldots, a_n b_n$ are not in AP.

(d) $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots, \frac{a_n}{b_n}$ are not in AP.

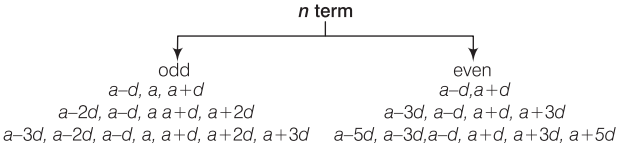
(iv) If $a_1, a_2, \ldots, a_n$ are in AP, then
$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots$
i.e. in an AP, the sum of two terms equidistant from the beginning and end is constant and is equal to sum of first and last terms.

(v) If $a_1, a_2, a_3, a_4, \ldots, a_n$ are in AP, then terms taken in regular intervals form a sequence are also in AP.

(vi) Any term in AP can be expressed as arithmetic mean of the terms equidistant from it, except first term, i.e. $a_r = \frac{a_{r-k} + a_{r+k}}{2}$.

(vii) If a is first term, d is common difference of an AP having m elements, then nth term from the end is $(m - n + 1)$th term from the beginning. Thus, nth term from the end $= a + (m - n)d$

(viii) In case we have to make selection of few terms which are in AP, then suggested choice is as follows (specially when sum of terms is given).



Arithmetic Mean

1. Single Arithmetic Mean

A number 'A' is said to be the single arithmetic mean between two numbers 'a' and 'b' provided a, A, b are in AP and $A = \frac{a+b}{2}$.

2. n Arithmetic Mean between Two Numbers

Let $A_1, A_2, A_3, \ldots, A_n$ be the n arithmetic means between two numbers a and b, then $a, A_1, A_2, A_3, \ldots, A_n, b$ are in AP.

$$A_k = a + kd = a + k\left(\frac{b-a}{n+1}\right), \text{ where } d = \frac{b-a}{n+1}$$

$n =$ The number of arithmetic mean and $k = 1, 2, 3, \ldots, n$.

Note (i) The sum of n arithmetic means between two given numbers is n times the single arithmetic mean between them,
i.e. if a and b are two numbers and $A_1, A_2, A_3, \ldots, A_n$ are n arithmetic means, then

$$A_1 + A_2 + A_2 + \ldots + A_n = n\left(\frac{a+b}{2}\right)$$

(ii) Let $a_1, a_2, a_3, \ldots, a_n$ be n positive real numbers (not all equals) and m be a real number, then

$$\left(\frac{a_1^m + a_2^m + \ldots + a_n^m}{n}\right) > \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m,$$

if $m \in R - [0, 1]$. However if $m \in (0, 1)$, then

$$\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} < \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m$$

REMEMBER

(i) If $a_p = q$ and $a_q = p$, then $a_{p+q} = 0, T_r = p + q - r$
(ii) If $pT_p = qT_q$, then $a_{p+q} = 0$
(iii) If $a_p = \frac{1}{q}$ and $a_q = \frac{1}{p}$, then $a_{pq} = 1$
(iv) If $S_p = q$ and $S_q = p$, then $S_{p+q} = -(p+q)$
(v) If $S_p = S_q$, then $S_{p+q} = 0$

Geometric Progression (GP)

A sequence of numbers (t_n) is said to be in geometric progression (GP), when the first term is non-zero and each successive term is r times the preceding term,
i.e. $\frac{t_n}{t_{n-1}} = r$ (constant) for all $n \in N, n \geq 2$, the constant value is called **common ratio**.
The standard form of GP is $a, ar, ar^2, \ldots, ar^{n-1}$.

nth Term of GP

If a is the first term and r is the common ratio, then

(i) nth term from beginning of a GP is $t_n = ar^{n-1}$

(ii) nth term from the end of a GP is

$$t_n' = \frac{l}{r^{n-1}} \qquad [\text{where, } l \text{ is the last term}]$$

Sum of First n Terms of a GP

(i) Sum of n terms of the series, $S_n = \begin{cases} \frac{a(r^n - 1)}{r - 1}, & \text{if } r \neq 1 \\ na, & \text{if } r = 1 \end{cases}$

(ii) Sum of infinite GP, $S_\infty = \begin{cases} \infty, & \text{if } r > 1 \\ \frac{a}{1-r}, & \text{if } |r| < 1 \end{cases}$

Note Infinite GP has definite value iff $|r| < 1$.

Properties of GP

(i) If $a_1, a_2, a_3, \ldots$ are in GP, then
(a) $a_1k, a_2k, a_3k, \ldots$ are also in GP.
(b) $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, \ldots$ are also in GP.
(c) $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots$ are also in GP.
(d) $a_1^k, a_2^k, a_3^k, \ldots$ are also in GP.

(ii) If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two GP's, then
(a) $a_1b_1, a_2b_2, a_3b_3, \ldots$ are also in GP.
(b) $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ are also in GP.
(c) $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3, \ldots$ are not in GP.

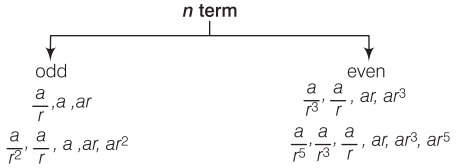
(iii) If $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ are in GP, the product of two terms which are equidistant from beginning and last is always constant and it is equal to product of first and last term, i.e. $a_1a_n = a_2a_{n-1} = a_3a_{n-2} = \ldots$

(iv) Each term of a GP can be expressed as product of square roots of the two terms which are equidistant from it, i.e. $a_r = \sqrt{a_{r-k} \cdot a_{r+k}}$.

(v) If $a_1, a_2, a_3, \ldots$ are in GP of positive terms, then $\log a_1, \log a_2, \log a_3, \ldots$ will be in AP and *vice-versa*.

(vi) If $a^{n_1}, a^{n_2}, a^{n_3}, \ldots, a^{n_m}$ are in GP, then $n_1, n_2, n_3, \ldots, n_m$ will form an AP.

(vii) In case we have to make selection of few terms which are in GP, then suggested terms are as follows



Geometric Mean

(i) If a and b are two numbers (either both positive or both negative), then geometric mean of these numbers is $G = \sqrt{ab}$. It is to be noted that the resultant sequence a, G, b is in GP.

(ii) If $a_1, a_2, a_3, \ldots, a_n$ are positive numbers, then the geometric mean of these numbers is given by

$$G = (a_1 \cdot a_2 \cdot a_3 \cdot a_4 \ldots a_n)^{1/n}$$

(iii) The numbers $G_1, G_2, G_3, \ldots, G_n$ are said to be geometric mean between a and b, if $a, G_1, G_2, G_3, \ldots, G_n, b$ are in GP, where

$$G_K = a\left(\frac{b}{a}\right)^{\frac{k}{n+k}}, \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} \text{ and } k = 0, 1, \ldots, n$$

Note (i) If $G_1, G_2, G_3, \ldots, G_n$ are n geometric means, then $G_1 \cdot G_2 \cdot G_3 \ldots G_n = (ab)^{n/2}$

(ii) If A and G are respectively arithmetic and geometric means between two positive numbers a and b, then

(a) AM $\geq$ GM $\Rightarrow A \geq G$

(b) The quadratic equation having a, b as roots, is $x^2 - 2Ax + G^2 = 0$ where, $a = A + \sqrt{A^2 - G^2}$, $b = A - \sqrt{A^2 - G^2}$

(iii) Product of GP

(a) If the number of terms is odd, then product of terms = (middle term)$^{\text{number of terms}}$

(b) If the number of terms is even, then product of terms = (geometric mean of middle term)$^{\text{number of terms}}$

REMEMBER

(i) If $a_p = x$ and $a_q = y$, then $a_n = \left(\frac{x^{n-q}}{y^{n-p}}\right)^{\frac{1}{p-q}}$.

(ii) If $a_{m+n} = p$ and $a_{m-n} = q$, then $a_m = \sqrt{pq}$ and $a_n = p\left(\frac{q}{p}\right)^{\frac{m}{2n}}$.

(iii) If a, b and c are the pth, qth and rth terms of a GP, then $a^{q-r} \times b^{r-p} \times c^{p-q} = 1$

(iv) nth term of $b + bb + bbb + \ldots$ is, $a_n = \frac{b}{9}(10^n - 1)$; $b = 1, 2, \ldots, 9$.

(v) nth term of $0.b + 0.bb + 0.bbb + \ldots$ is, $a_n = \frac{b}{9}(1 - 10^n)$; $b = 1, 2, \ldots, 9$

Harmonic Progression (HP)

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ of non-zero numbers is called a harmonic progression or a harmonic sequence, if the sequence $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}, \ldots$ is an arithmetic progression.

*n*th Term of HP

The nth of HP is the reciprocal of the nth term of the corresponding AP. If $a_1, a_2, a_3, \ldots, a_n$ are in HP and the common difference of the corresponding AP is d, then the nth term of the HP is given by $a_n = \frac{1}{1/a_1 + (n-1)d}$.

Harmonic Mean

If a and b are two non-zero numbers, then the harmonic mean of a and b is a number H such that the numbers a, H, b are in HP and $H = \frac{2}{1/a + 1/b} = \frac{2ab}{a+b}$.

Harmonic Mean of *n* Numbers

If $a_1, a_2, \ldots, a_n$ are n non-zero numbers, then the harmonic mean H of these numbers is given by $H = \frac{n}{1/a_1 + 1/a_2 + \ldots + 1/a_n}$.

Insertion of Harmonic Means between *a* and *b*

The n numbers $H_1, H_2, \ldots, H_n$ are said to be n harmonic means between a and b, if $a, H_1, H_2, H_3, \ldots, H_n, b$ are in HP.

i.e. if $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \ldots, \frac{1}{H_n}, \frac{1}{b}$ are in AP.

$$\Rightarrow \frac{1}{H_n} = \frac{1}{a} + \frac{n(a-b)}{(n+1)ab} = \frac{b+na}{ab(n+1)}$$

REMEMBER

(i) If A, G and H are respectively AM, GM and HM between a and b both being unequal and positive, then

(a) $G^2 = AH$ i.e. A, G and H are in geometric progression

(b) $A \geq G \geq H$ equality holds only when all terms are equal

(ii) If A, G and H are arithmetic, geometric and harmonic means between three given numbers a, b and c, then the equation having a, b and c as its roots, is $x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$.

Special Series

Sometimes, the given series is neither an AP nor a GP. But after subtraction of consecutive terms of this series, we obtain an AP or GP, such process is called method of difference which is discussed below.

Method of Difference

Suppose $a_1, a_2, a_3, \ldots$ is a sequence such that the sequence $(a_2 - a_1), (a_3 - a_2) \ldots$ is either an AP or GP. The nth term of this sequence can be obtained with the help of following steps.

Step I Denote the nth term and sum of the series upto n terms by a_n and S_n respectively, such that

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n$$

Step II Rewrite the given series with each term shifted by one place to the right,

i.e. $$S_n = 0 + a_1 + a_2 + a_3 + \ldots + a_n$$

Step III Subtract the above two series to find a_n

$$0 = a_1 + (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_n - a_{n-1}) - a_n$$

$$\Rightarrow a_n = a_1 + (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_n - a_{n-1})$$

Step IV Find the sum of the series $S_n = \sum_{k=1}^{n} a_k$.

Arithmetico-Geometric Progression (AGP)

If a is the first term, d is the common difference and r is the common ratio, then

$$a, (a + d)r, (a + 2d)r^2, \ldots, [a + (n-1)d]r^{n-1}$$

form an AGP

(i) nth term of an AGP, $T_n = [a + (n-1)d]r^{n-1}$

(ii) The sum of n terms of an AGP is

$$S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a + (n-1)d]r^n}{1-r}$$

(iii) Sum of infinite AGP, $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

Note (i) $1 + 2 + 3 + \ldots + n = \Sigma n = \frac{n(n+1)}{2}$

(ii) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

(iv) $1 + 3 + 5 + \ldots + (2n-1) = n^2$

(v) $2 + 4 + 6 + \ldots + 2n = n(n+1)$

(vi) $1 + x + x^2 + \ldots + \infty = (1-x)^{-1}, -1 < x < 1$

(vii) $1 + 2x + 3x^2 + 4x^3 + \ldots + \infty = (1-x)^{-2}, -1 < x < 1$

(viii) To obtain the sum $\sum_{i<j} a_i a_j$, we use identity

$$2\sum_{i<j} a_i a_j = (a_1 + a_2 + a_3 + \ldots + a_n)^2 - (a_1^2 + a_2^2 + \ldots + a_n^2)$$

(ix) If $s = \frac{1}{a_1 a_2 \ldots a_r} + \frac{1}{a_2 a_3 \ldots a_{r+1}} + \ldots + \frac{1}{a_n a_{n+1} \ldots a_{n+r-1}}$,

then $S = \frac{1}{(r-1)(a_2 - a_1)}\left[\frac{1}{a_1 \cdot a_2 \ldots a_{r-1}} - \frac{1}{a_{n+1} \cdot a_{n+2} \ldots a_{n+r-1}}\right]$

Particular Cases

(a) $\frac{1}{a_1 \cdot a_2} + \frac{1}{a_2 \cdot a_3} + \ldots + \frac{1}{a_n \cdot a_{n+1}} = \frac{n}{a_1 a_{n+1}}$

(b) $\frac{1}{a_1 a_2 a_3} + \frac{1}{a_2 a_3 a_4} + \frac{1}{a_3 a_4 a_5} + \ldots + \frac{1}{a_n a_{n+1} a_{n+2}}$

$$= \frac{1}{2(a_2 - a_1)}\left[\frac{1}{a_1 a_2} - \frac{1}{a_{n+1} a_{n+2}}\right]$$

(x) If $S = a_1 a_2 \ldots a_r + a_2 a_3 \ldots a_{r+1} + a_3 a_4 \ldots a_{r+2} + \ldots + a_n a_{n+1} \ldots a_{n+r-1}$

$$S = \frac{1}{(r+1)(a_2 - a_1)}[a_n \cdot a_{n+1} \ldots a_{n+r} - a_0 a_1 \ldots a_r]$$

$[\because a_0 = a_1 - d]$

Particular Cases

(a) $a_1 a_2 + a_2 a_3 + a_3 a_4 + \ldots + a_n a_{n+1}$

$$= \frac{1}{3(a_2 - a_1)}[a_n a_{n+1} a_{n+2} - a_0 a_1 a_2]$$

(b) $a_1 a_2 a_3 + a_2 a_3 a_4 + a_3 a_4 a_5 + \ldots + a_n a_{n+1} a_{n+2}$

$$= \frac{1}{4(a_2 - a_1)}[a_n a_{n+1} a_{n+2} a_{n+3} - a_0 a_1 a_2 a_3]$$

MASTER STROKES

1. If $a_1, a_2, a_3, \ldots, a_{n+1}$ are in AP, then $\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \ldots \frac{1}{a_n a_{n+1}}$ is equal to

(a) $\frac{n-1}{a_1 a_{n+1}}$ (b) $\frac{1}{a_1 a_{n+1}}$

(c) $\frac{n+1}{a_1 a_{n+1}}$ (d) $\frac{n}{a_1 a_{n+1}}$

2. If a_n is the nth term of an AP and $a_7 = 15$, then the value of the common difference that would make $a_2\ a_7\ a_{12}$ greatest, is

(a) 9 (b) $\frac{9}{4}$ (c) 0 (d) 18

3. If $\log_5 2$, $\log_5 2^x - 3$ and $\log_5\left(\frac{17}{2} + 2^{x-1}\right)$ are in AP, then the value of x is

(a) 0 (b) –1 (c) 5 (d) 3

4. If the fourth power of the common difference of an AP with integer entries is added to the product of any four consecutive terms of it, then the resulting sum is

(a) an even integer (b) an odd integer

(c) the square of an integer (d) the cube of an integer

5. The first and last term of an AP are a and l, respectively. If s is the sum of all the terms of the AP, then the common difference is

(a) $\frac{l^2 - a^2}{2s - (l + a)}$ (b) $\frac{l^2 - a^2}{2s - (l - a)}$

(c) $\frac{l^2 + a^2}{2s + (l + a)}$ (d) $\frac{l^2 + a^2}{2s - (l + a)}$

6. Let S_n denotes the sum of first n terms of an AP. If $S_{2n} = 3S_n$, then the ratio of $\frac{S_{3n}}{S_n}$ is equal to

(a) 4 (b) 6 (c) 8 (d) 10

7. Let the sequence $a_1, a_2, a_3, \ldots, a_n$ form an AP. Then, $a_1^2 - a_2^2 + a_3^2 - a_4^2 + \ldots + a_{2n-1}^2 - a_{2n}^2$ is equal to

(a) $\frac{n}{2n-1}(a_1^2 - a_{2n}^2)$ (b) $\frac{2n}{n-1}(a_{2n}^2 - a_1^2)$

(c) $\frac{n}{n+1}(a_1^2 + a_{2n}^2)$ (d) None of these

8. Let $a_1, a_2, a_3, \ldots$ be n terms of an AP. If $\frac{a_1 + a_2 + \ldots + a_p}{a_1 + a_2 + \ldots + a_q} = \frac{p^2}{q^2}$, $p \neq q$, then $\frac{a_6}{a_{21}}$ equals

(a) $\frac{41}{11}$ (b) $\frac{7}{2}$ (c) $\frac{2}{7}$ (d) $\frac{11}{41}$

9. If $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the AM between a and b, then the value of n is

(a) 0 (b) 1

(c) –1 (d) None of these

10. Let $a_1, a_2, a_3, \ldots, a_n$ be a GP such that $\frac{a_4}{a_6} = \frac{1}{4}$ and $a_2 + a_5 = 216$. Then, a_1 is equal to

(a) 12 or $\frac{108}{7}$ (b) 10

(c) 7 or $\frac{54}{7}$ (d) None of these

11. If $a_1, a_2, a_3 (a_1 > 0)$ are three successive terms of a GP with common ratio r, the value of r for which $a_3 > 4a_2 - 3a_1$ holds, is given by

(a) $1 < r < 3$ (b) $-3 < r < -1$

(c) $r > 3$ or $r < 1$ (d) None of these

12. If the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$ are in GP, then

(a) $c^3a = b^3d$ (b) $ca^3 = bd^3$

(c) $a^3b = c^3d$ (d) $ab^3 = cd^3$

13. If $|\alpha| < 1, |\beta| < 1$

$1 - \alpha + \alpha^2 - \alpha^3 + \ldots$ to $\infty = S_1$

$1 - \beta + \beta^2 - \beta^3 + \ldots$ to $\infty = S_2$, then

$1 - \alpha\beta + \alpha^2\beta^2 - \alpha^3\beta^3 + \ldots$ to ∞ equals

(a) S_1S_2 (b) $\frac{S_1S_2}{1 + S_1S_2}$

(c) $\frac{S_1S_2}{1 - S_1 - S_2 + 2S_1S_2}$ (d) $\frac{1}{1 + S_1S_2}$

14. Let a_n be the nth term of the GP of positive numbers. Let $\sum_{n=1}^{100} a_{2n} = \alpha$ and $\sum_{n=1}^{100} a_{2n-1} = \beta$, such that $\alpha \neq \beta$, then the common ratio is

(a) $\frac{\alpha}{\beta}$ (b) $\frac{\beta}{\alpha}$ (c) $\sqrt{\frac{\alpha}{\beta}}$ (d) $\sqrt{\frac{\beta}{\alpha}}$

15. If a is the AM of b and c and the two geometric means are G_1 and G_2 are inserted between b and c such that $G_1^3 + G_2^3 = \lambda abc$, then λ equals

(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) 3

16. If $x > 1$, $y > 1$, $z > 1$ are in GP, then $\frac{1}{1 + \ln x}$, $\frac{1}{1 + \ln y}$, $\frac{1}{1 + \ln z}$ are in

(a) AP (b) HP

(c) GP (d) None of these

17. If $x^2 + 9y^2 + 25z^2 = xyz\left(\frac{15}{x} + \frac{5}{y} + \frac{3}{z}\right)$, then x, y and z are in

(a) AP (b) GP (c) AGP (d) HP

18. If a, b, c are in HP, then

(a) $a^n + c^n > b^n$ (b) $a^n + c^n > 2b^n$

(c) $a^n + b^n > 2c^n$ (d) $b^n + c^n > 2a^n$

19. The sum to infinity of the series $1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \ldots$ is

(a) 2 (b) 3 (c) 4 (d) 6

20. The sum of all possible products of the first n natural numbers taken two by two is

(a) $\frac{1}{24}n(n+1)(n-1)(3n+2)$

(b) $\frac{n(n+1)(2n+1)}{6}$

(c) $\frac{n(n+1)(n-1)(2n+3)}{24}$

(d) None of the above

21. If $t_n = \frac{1}{4}(n+2)(n+3)$ for $n = 1, 2, 3, \ldots$, then $\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \ldots + \frac{1}{t_{2003}}$ equals

(a) $\frac{4006}{3006}$ (b) $\frac{4003}{3007}$ (c) $\frac{4006}{3008}$ (d) $\frac{4006}{3009}$

22. The limiting value of the sum to n terms of the series $\frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \ldots$ as $n \to \infty$ is

(a) 0 (b) 2 (c) $\frac{1}{2}$ (d) 1

23. If 1, $\log_9(3^{1-x} + 2)$ and $\log_3(4 \cdot 3^x - 1)$ are in AP, then x is equal to

(a) $\log_4 3$ (b) $\log_3 4$

(c) $1 - \log_3 4$ (d) $\log_3 0.25$

24. If three positive real numbers a, b, c are in AP such that $abc = 4$, then the minimum value of b is

(a) $2^{1/3}$ (b) $2^{2/3}$ (c) $2^{1/2}$ (d) $2^{3/2}$

25. Let $S \subset (0, \pi)$ denotes the set of values of x satisfying the equation $8^{1 + |\cos x| + \cos^2 x + |\cos^3 x| + \ldots \text{ to } \infty} = 4^3$. Then, S equals

(a) $\left\{\frac{\pi}{3}\right\}$ (b) $\left\{\frac{\pi}{3}, -\frac{2\pi}{3}\right\}$ (c) $\left\{-\frac{\pi}{3}, \frac{2\pi}{3}\right\}$ (d) $\left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$

26. If the sides of a triangle are in GP and its larger angle is twice the smallest, then the common ratio r satisfies the inequality
(a) $0 < r < \sqrt{2}$ (b) $1 < r < \sqrt{2}$
(c) $1 < r < 2$ (d) None of these

27. If $b-c$, $2b-x$ and $b-a$ are in HP, then $a-(x/2)$, $b-(x/2)$ and $c-(x/2)$ are in
(a) AP (b) GP
(c) HP (d) None of these

28. The sum to n terms of the series $\frac{3}{1^2}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^2}+\ldots$, is
(a) $\frac{6n}{n+1}$ (b) $\frac{9n}{n+1}$ (c) $\frac{12n}{n+1}$ (d) $\frac{3n}{n+1}$

29. Let $a_1, a_2, \ldots, a_{10}$ be in AP and $h_1, h_2, \ldots, h_{10}$ be in HP. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then $a_4 h_7$ equals
(a) 2 (b) 3 (c) 5 (d) 6

30. Let $S_1, S_2, \ldots$ be squares such that for each $n \geq 1$, the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm, then for which of the following values of n is the area of S_n less than 1 sq cm?
(a) 7 (b) 8 (c) 5 (d) 6

31. Every term of a GP is positive and also every term is the sum of two preceding terms. Then, the common ratio of the GP is
(a) $\frac{1-\sqrt{5}}{2}$ (b) $\frac{\sqrt{5}+1}{2}$ (c) $\frac{\sqrt{5}-1}{2}$ (d) 1

32. Let $S_K = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{(k+1)^i}$. Then, $\sum_{k=1}^{n} k s_k$ is equal to
(a) $\frac{n(n+1)}{2}$ (b) $\frac{n(n-1)}{2}$ (c) $\frac{n(n+2)}{2}$ (d) $\frac{n(n+3)}{2}$

33. If $\sum_{r=1}^{n} T_r = \frac{n(n+1)(n+2)(n+3)}{8}$, then $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{T_r}$ equals
(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{8}$

34. If $\sum_{r=1}^{n} r$, $\frac{\sqrt{10}}{3}\sum_{r=1}^{n} r^2$, $\sum_{r=1}^{n} r^3$ are in GP, then the value of n is
(a) 2 (b) 3 (c) 4 (d) 5

35. If $a_1, a_2, a_3, \ldots, a_n$ are in HP and $f(k) = \sum_{r=1}^{n} a_r - a_k$, then $\frac{a_1}{f(1)}, \frac{a_2}{f(2)}, \ldots, \frac{a_n}{f(n)}$ are in
(a) AP (b) GP
(c) HP (d) None of these

36. In a sequence of $(4n+1)$ terms, the first $(2n+1)$ terms are in AP whose common difference is 2 and the last $(2n+1)$ terms are in GP whose common ratio is 0.5. If the middle terms of the AP and GP are equal, then the middle term of the sequence is
(a) $\frac{n\cdot 2^{n+1}}{2^n-1}$ (b) $\frac{n\cdot 2^{n+1}}{2^{2n}-1}$ (c) $n\cdot 2^n$ (d) $n\cdot 2^{n+1}$

37. If S_K denotes the sum of first K terms of a GP. Then, S_n, $S_{2n}-S_n$, $S_{3n}-S_n$ are in
(a) AP (b) GP
(c) HP (d) None of these

38. In the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, …, where n consecutive terms have the value n, the 150th term is
(a) 17 (b) 16
(c) 18 (d) None of these

39. The value of $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1$ is
(a) Σn (b) Σn^2
(c) Σn^3 (d) None of these

40. The sum of the products of $2n$ numbers $\pm 1, \pm 2, \pm 3, \ldots, \pm n$ taking two at a time is
(a) $-\sum_{r=1}^{n} r$ (b) $\sum_{r=1}^{n} r^2$
(c) $-\sum_{r=1}^{n} r^2$ (d) None of these

41. If $a_1, a_2, a_3, \ldots, a_n$ are non-zero real numbers such that $(a_1^2+a_2^2+\ldots a_{n-1}^2)(a_2^2+a_3^2+\ldots+a_n^2) \leq (a_1a_2+a_2a_3+\ldots+a_{n-1}a_n)^2$, then $a_1, a_2, \ldots, a_n$ are in
(a) HP (b) GP
(c) AP (d) None of these

42. Let A_1, G_1 and H_1 denote the arithmetic, geometric and harmonic means respectively of two distinct positive numbers. For $n \geq 2$, let A_{n-1}, G_{n-1} and H_{n-1} have arithmetic, geometric and harmonic means as A_n, G_n and H_n, respectively. Then, which of the following statements is correct?
(a) $G_1 > G_2 > G_3 > \ldots$
(b) $G_1 < G_2 < G_3 < \ldots$
(c) $G_1 = G_2 = G_3 = \ldots$
(d) $G_1 < G_3 < G_5 \ldots$ and $G_2 > G_4 > G_6 > \ldots$

43. Let $A_n = \left(\frac{3}{4}\right) - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \ldots + (-1)^{n-1}\left(\frac{3}{4}\right)^n$ and $B_n = 1 - A_n$. The least positive integer number satisfying $B_n > A_n$ for all $n \geq n_0$, is
(a) 5 (b) 6 (c) 7 (d) 8

44. Let S_K, $K = 1, 2, \ldots, 100$, denotes the sum of the infinite geometric series whose first term is $\frac{K-1}{K!}$ and the common ratio is $\frac{1}{K}$. Then, the value of $\frac{100^2}{100!} + \sum_{K=1}^{100} |(K^2-3K+1)S_K|$ is
(a) 3 (b) 6 (c) 8 (d) 9

45. Let $a_1, a_2, a_3, \ldots, a_{11}$ be real numbers satisfying $a_1 = 15$, $27 - 2a_2 > 0$ and $a_K = 2a_{K-1} - a_{K-2}$ for $K = 3, 4, \ldots, 11$. If $\frac{a_1^2 + a_2^2 + \ldots + a_{11}^2}{11} = 90$, then the value of $\frac{a_1 + a_2 + \ldots + a_{11}}{11}$ is equal to

(a) 0 *(b)* 1 *(c)* 2 *(d)* 9

46. A person is to count 4500 currency notes. Let a_n denotes the number of notes he counts in the nth minute. If $a_1 = a_2 = \ldots = a_{10} = 150$ and $a_{10}, a_{11}, \ldots$ are in AP with common difference –2, then the time taken by him to count all notes is

(a) 125 min *(b)* 135 min
(c) 24 min *(d)* 34 min

47. If m is the AM of two distinct real numbers l and n ($l, n > 1$) and G_1, G_2, G_3 are three geometric means between l and n, then $G_1^4 + 2G_2^4 + G_3^4$ equals

(a) $4\,lmn^2$ *(b)* $4\,l^2m^2n^2$ *(c)* $4\,l^2mn$ *(d)* $4\,lm^2n$

48. The sum of first 9 terms of the series $\frac{1^3}{1} + \frac{1^3 + 2^3}{1 + 3} + \frac{1^3 + 2^3 + 3^3}{1 + 3 + 5} + \ldots$ is

(a) 142 *(b)* 192 *(c)* 71 *(d)* 96

49. The sum of the series $1 + \frac{4}{3} + \frac{10}{9} + \frac{28}{27} + \ldots$ upto n terms is

(a) $n - \frac{1}{3} + \frac{1}{3 \cdot 2^{n-1}}$ *(b)* $\frac{7}{6}n + \frac{1}{6} - \frac{1}{3 \cdot 2^{n-1}}$
(c) $\frac{5}{3}n - \frac{7}{6} + \frac{1}{2 \cdot 3^{n-1}}$ *(d)* $n + \frac{1}{2} - \frac{1}{3 \cdot 2^{n-1}}$

50. If $\frac{48}{2 \cdot 3} + \frac{47}{3 \cdot 4} + \frac{46}{4 \cdot 5} + \ldots + \frac{2}{48 \cdot 49} + \frac{1}{49 \cdot 50} = \frac{51}{2} + k\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{50}\right)$, then k equals

(a) 2 *(b)* –1 *(c)* $-\frac{1}{2}$ *(d)* 1

51. Let $a_1, a_2, a_3, \ldots$ be a harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$, is

(a) 22 *(b)* 23 *(c)* 24 *(d)* 25

52. $\log_{10} 2$, $\log_{10}(2^x - 1)$ and $\log_{10}(2^x + 3)$ are three consecutive terms of an AP for

(a) more than two real x *(b)* no real x
(c) exactly one real x *(d)* exactly two real x

53. Let a_n be the nth term of an AP. If $\sum_{r=1}^{100} a_{2r} = \alpha$ and $\sum_{r=1}^{100} a_{2r-1} = \beta$, then the common difference of the AP is

(a) $\frac{\alpha - \beta}{100}$ *(b)* $\beta - \alpha$ *(c)* $\frac{\alpha - \beta}{200}$ *(d)* $\alpha - \beta$

54. Let $a_1, a_2, a_3, \ldots, a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i-1}^{p} a_i$, $1 \le p \le 100$. For any integer n with $1 \le n \le 20$, let $m = 5n$. If S_m / S_n does not depend on n, then a_2 is

(a) 9 *(b)* 8 *(c)* 7 *(d)* 5

55. Let the harmonic mean of two positive real numbers a and b be 4. If q is a positive real number such that $a, 5, q, b$ is an arithmetic progression, then the value(s) of $|q - a|$ is/are

(a) 3, 4 *(b)* 2, 5 *(c)* 3, 6 *(d)* 6, 9

56. Suppose all the terms of an arithmetic progression (AP) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this AP is

(a) 5 *(b)* 6 *(c)* 8 *(d)* 9

57. Let a, b, c be positive integers such that b/a is an integer. If a, b, c are in geometric progression and the arithmetic mean of a, b, c is $b + 2$, then the value of $\frac{a^2 + a - 14}{a + 1}$ is

(a) 2 *(b)* 4 *(c)* 6 *(d)* 8

58. Three positive numbers form an increasing GP. If the middle term in the GP is doubled, the new numbers are in AP. Then, the common ratio of the GP is

(a) $2 - \sqrt{3}$ *(b)* $2 + \sqrt{3}$ *(c)* $\sqrt{2} + \sqrt{3}$ *(d)* $3 + \sqrt{2}$

59. Let $S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2$. Then, S_n can take the value(s)

(a) 1056 and 1332 *(b)* 1056 and 1088
(c) 1120 and 1332 *(d)* 1332 and 1432

60. If $(10)^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + \ldots + 10(11)^9 = k(10)^9$, then k is equal to

(a) 100 *(b)* 110 *(c)* $\frac{121}{10}$ *(d)* $\frac{441}{100}$

Answers

1. (d)	**2.** (c)	**3.** (d)	**4.** (c)	**5.** (a)
6. (b)	**7.** (a)	**8.** (d)	**9.** (b)	**10.** (a)
11. (c)	**12.** (a)	**13.** (c)	**14.** (a)	**15.** (b)
16. (b)	**17.** (d)	**18.** (b)	**19.** (b)	**20.** (a)
21. (d)	**22.** (d)	**23.** (c)	**24.** (b)	**25.** (d)
26. (b)	**27.** (b)	**28.** (a)	**29.** (d)	**30.** (b)
31. (b)	**32.** (d)	**33.** (b)	**34.** (c)	**35.** (c)
36. (a)	**37.** (b)	**38.** (a)	**39.** (d)	**40.** (c)
41. (b)	**42.** (c)	**43.** (b)	**44.** (a)	**45.** (a)
46. (d)	**47.** (d)	**48.** (d)	**49.** (d)	**50.** (b)
51. (d)	**52.** (c)	**53.** (a)	**54.** (a)	**55.** (b)
56. (c)	**57.** (c)	**58.** (b)	**59.** (a)	**60.** (a)

MATHEMATICAL INDUCTION

The word induction means the generalisation from particular cases or facts. In algebra or in other disciplines of Mathematics, there are certain results or statements that are formulated in terms of n, where n is a positive integer. To prove such statements, the well suited principle which is based on specific technique is the principle of mathematical induction.

Statements

A sentence or description which can be judged to be true or false, is called statements.

Mathematical Statements

Statements involving mathematical relations are known as the mathematical statements.

Peano's Theorem

The statement or proposition $P(n)$ is assumed to be true for all values of $n \in N$

(a) $P(1)$ is true (b) $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Principle of Mathematical Induction

Let $P(n)$ be a statement involving all natural numbers n such that

(i) The statement is true for $n = 1$, i.e. $P(1)$ is true.

(ii) If the statement is true for $n = k$, then the statement is also true for $n = k + 1$, i.e. truth of $P(k)$ = truth of $P(k+1)$, where k is some positive integer.

Then, $P(n)$ is true for all natural numbers n.

In order to prove the validity of the statements, we use following steps

***Step* I Verification step** Actual verification of the proposition for the starting value '1'.

***Step* II Induction step** Assuming the proposition to be true for 'k', $k \geq 1$ and then prove that it is true for the value $(k+1)$ which is the next higher integer.

***Step* III Generalisation step** Combine the above two steps.

Some Useful Results

Following results will be very helpful in answering objective type problems on mathematical induction.

(i) $1 + 2 + \ldots + n = \Sigma n = \dfrac{n(n+1)}{2}$

(ii) $1^2 + 2^2 + \ldots + n^2 = \Sigma n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(iii) $1^3 + 2^3 + \ldots + n^3 = \Sigma n^3 = \left\{\dfrac{n(n+1)}{2}\right\}^2$

(iv) $1^4 + 2^4 + \ldots + n^4 = \Sigma n^4 = \dfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

(v) The sum of first n even natural numbers $= \Sigma 2n = n(n+1)$

(vi) The sum of first n odd natural numbers $= \Sigma (2n-1) = n^2$

(vii) $\Sigma n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

(viii) $\Sigma n(n+1)(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$

(ix) $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + xy^{n-2} + y^{n-1})$ for all $n \in N$

(x) $x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 + \ldots - xy^{n-2} + y^{n-1})$ when n is an odd natural number.

MASTER STROKES

1. The sum of the terms in the nth bracket of the series $1 + (2+3+4) + (5+6+7+8+9) + \ldots$, is

(a) $(n-1)^3 + n^3$ (b) $(n+1)^3 + 8n^2$

(c) $\dfrac{(n+1)(n+2)}{6n}$ (d) $(n+1)^3 + n^3$

2. For all $n \in N$, $2 \cdot 4^{2n+1} + 3^{3n+1}$ is divisible by

(a) 7 (b) 5 (c) 11 (d) 209

3. $\dfrac{3}{4} + \dfrac{15}{16} + \dfrac{63}{64} + \ldots$ to n terms is equal to

(a) $n - \dfrac{4^n}{3} - \dfrac{1}{3}$ (b) $n + \dfrac{4^{-n}}{3} - \dfrac{1}{3}$ (c) $n + \dfrac{4^n}{3} - \dfrac{1}{3}$ (d) $n - \dfrac{4^{-n}}{3} + \dfrac{1}{3}$

4. For all odd positive integers n, the number $n(n^2-1)$ is divisible by

(a) 36 (b) 9

(c) 16 (d) 24

5. $\dfrac{\frac{1}{2}\cdot\frac{2}{2}}{1^3} + \dfrac{\frac{2}{2}\cdot\frac{3}{2}}{1^3+2^3} + \dfrac{\frac{3}{2}\cdot\frac{4}{2}}{1^3+2^3+3^3} + \ldots$ upto n terms is equal to

(a) $\dfrac{n-1}{n}$ (b) $\dfrac{n}{n+1}$

(c) $\dfrac{n+1}{n+2}$ (d) $\dfrac{n+1}{n}$

6. $\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\frac{1}{7\cdot9}+\ldots$ to n terms is equal to

(a) $\frac{n}{3(2n+3)}$ (b) $\frac{n}{(2n+3)}$

(c) $\frac{1}{(n+2)(n+4)}$ (d) None of these

7. For $n\in N$, $x^{n+1}+(x+1)^{2n-1}$ is divisible by

(a) x (b) $x+1$

(c) x^2+x+1 (d) x^2-x+1

8. The product of three consecutive natural numbers is divisible by

(a) 3 (b) 8 (c) 6 (d) 11

9. Let $S(K)=1+3+5+\ldots+(2K-1)=3+K^2$

Then, which of the following is true?

(a) Principle of mathematical induction can be used to prove the formula

(b) $S(K)\Rightarrow S(K+1)$

(c) $S(K)\not\Rightarrow S(K+1)$

(d) $S(1)$ is correct

10. For all $n\in N$, $\cos\theta\cos2\theta\cos4\theta\ldots\cos2^{n-1}\theta$ equals

(a) $\frac{\sin^n\theta}{2^n\sin\theta}$ (b) $\frac{\sin2^n\theta}{\sin\theta}$ (c) $\frac{\cos2^n\theta}{2^n\cos2\theta}$ (d) $\frac{\cos2^n\theta}{2^n\sin\theta}$

11. For all $n\in N$, Σn

(a) $<\frac{(2n+1)^2}{8}$ (b) $>\frac{(2n+1)^2}{8}$

(c) $=\frac{(2n+1)^2}{8}$ (d) None of these

12. $\frac{(n+2)!}{(n-1)!}$ is divisible by

(a) 6 (b) 11 (c) 24 (d) 26

13. For all $n\in N$, $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}$ is

(a) equal to $\sqrt{n}$

(b) less than or equal to $\sqrt{n}$

(c) greater than or equal to $\sqrt{n}$

(d) None of the above

14. If $3+5+9+17+33+\ldots$ to n terms $=2^{n+1}+n-2$, then nth term of LHS is

(a) 3^n-1 (b) $2n+1$

(c) 2^n+1 (d) $3n-1$

15. For all $n\in N$, $\frac{n^5}{5}+\frac{n^3}{3}+\frac{7}{15n}$, is

(a) an integer (b) a natural number

(c) a positive fraction (d) None of these

16. If $U_{n+1}=3U_n-2U_{n-1}$ and $U_0=2$, $U_1=3$, then U_n is equal to

(a) $1-2^n$ (b) 2^n+1

(c) 2^n-1 (d) 2^n+2

Answers

1. (a)	2. (c)	3. (b)	4. (d)	5. (b)
6. (a)	7. (c)	8. (c)	9. (b)	10. (a)
11. (a)	12. (a)	13. (b)	14. (c)	15. (b)
16. (b)				

PERMUTATIONS AND COMBINATIONS

PERMUTATION

Permutation is used to calculate the number of ways of arrangement of some objects.

Factorial Notation

Let n be a positive integer, then continued product of first n natural numbers is called factorial n and is denoted by $n!$ or $\lfloor\underline{n}$.

$\therefore\quad n!=n(n-1)(n-2)\ldots3\cdot2\cdot1$

Note (i) $0!=1$ (ii) $n!=n(n-1)!$

Fundamental Principles of Counting

There are two fundamental principles of counting.

(i) **Multiplication Rule**

If a task can be done by m different ways and in continuation other task can be done by n different ways. Then, total number of different ways of simultaneous completion of the task is $m\times n$.

or

If a job has n parts and job will be completed only when each part get completed. Let first part can be completed in p_1 ways, the second can be completed in p_2 ways ,..., nth part can be completed in p_n ways. Then, total number of ways of doing the job is $p_1\times p_2\times p_3\times\ldots\times p_n$.

(ii) **Addition Rule**

If a task can be completed in m different ways and another task can be completed in n different ways. Then, exactly one of the task can be completed by $(m+n)$ possible ways.

Do You Know

In 2010 on World Maths Day, 1.13 million students from more than 235 countries set a record answering 479732613 questions correctly.

REMEMBER

(i) All the numbers whose last digit is an even number 0, 2, 4, 6 or 8, are divisible by 2.
(ii) If sum of the digits is divisible by 3, then number is divisible by 3 and if sum is divisible by 9, then number is divisible by 9.
(iii) If last two digits are divisible by 4, then the number is divisible by 4.
(iv) If last two digits are divisible by 25, then number is divisible by 25.
(v) If a number ends with 0 and 5, then number is always divisible by 5.
(vi) If a number is divisible by 2 and 3, then the number is divisible by 6.
(vii) If last three digits of a number are divisible by 8, then the number is divisible by 8.

Permutation

Permutation is arrangement of objects. Basically, it involves two steps, i.e. selection and then arrangement. When we arrange the objects, we are concern about their order. e.g. Arrangement of the letters a, b, c taking two at a time are ab, bc, ca, ba, cb, ac.

Meaning of ${}^nP_r$

Number of permutations of n distinct objects taking r at a time is denoted by ${}^nP_r$.

$${}^nP_r = \frac{n!}{(n-r)!}, \forall\ 0 \le r \le n$$

$$= n\,(n-1)\,(n-2)\ldots(n-r+1),\ \forall\ n \in N \text{ and } r \in W$$

Properties of ${}^nP_r$

(i) ${}^nP_0 = 1,\ {}^nP_1 = n$ and ${}^nP_{n-1} = n!$
(ii) ${}^nP_r = n \cdot {}^{n-1}P_{r-1} = r \cdot {}^{n-1}P_{r-1} + {}^{n-1}P_r$
(iii) ${}^{n-1}P_r = (n-r)\ {}^{n-1}P_{r-1}$

Different Cases of Permutation

(i) The number of permutations of n objects taken r at a time when repetition is not allowed, is

$${}^nP_r = \frac{n!}{(n-r)!}$$

(ii) The number of permutations of n distinct objects taken all at a time is ${}^nP_n = n!$.
(iii) The number of permutations of n different objects taken r at a time when repetition of objects are allowed, is equal to n^r.

Permutation of Objects not all Distinct

Number of arrangements of n objects when p particular objects are identical of one kind, q particular objects are identical of other kind and r particular objects are identical of third kind, such that $p + q + r = n$, then

$$\text{Number of permutations} = \frac{n!}{p!\,q!\,r!}$$

In general, the number of permutations of n things, of which p_1 are alike of one kind, p_2 are alike of second kind, p_3 are alike of third kind, … , p_r are alike of rth kind such that

$$p_1 + p_2 + p_3 + \ldots + p_r = n, \text{ is } \frac{n!}{p_1!\,p_2!\,p_3!\ldots p_r!}.$$

SPEED BOOSTER

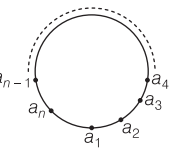
(i) If we are given n different digits $(n_1, n_2, \ldots, n_n)$, then sum of the digits in the unit place of all numbers formed without repetition is $(n-1)!(n_1 + n_2 + \ldots + n_n)$.
(ii) Sum of the total numbers formed by n digits $(n_1, n_2, n_3, \ldots, n_n)$ is $(n-1)!(n_1 + n_2 + n_3 + \ldots + n_n)\,(111\ldots n \text{ times})$.
(iii) The number of ways in which n (one type of different things) and m (another type of different things) can be arranged in a row alternatively is $2m\,n!$, here $(m = n)$.

Permutations under Restrictions

(i) If we place n objects in a row, then number of positions (gaps) are $(n+1)$ when question is asked to made an alternate arrangement, then place one kind of object and then select the gap to place another kind of objects.
(ii) Number of permutations of n different things taken r at a time, when particular things are to be always included in each arrangement, is $r \cdot {}^{n-1}P_{r-1}$.
(iii) Number of permutations of n different things taken r at a time, when s particular things are to be always included in each arrangement, is $s\,[r-(s-1)]^n\,{}^sP_{r-s}$.
(iv) Number of permutations of n different things taken r at a time, when particular things are never taken in each arrangement is ${}^{n-1}P_r$.
(v) Number of permutations of n different things taken all at a time, when m specified things always come together, is $m! \times (n-m+1)!$.
(vi) Number of permutations of n different things taken all at a time, when m specified things never come together, is $n! - m!(n-m+1)!$.

Circular Permutation

The number of circular permutations of n different things taken all at a time is $(n-1)!$.



Note

(i) If anti-clockwise and clockwise order of arrangement are not distinct (e.g. arrangement of beads in a necklace, arrangement of flowers in garland etc.), then number of circular permutations, n different things taken all at a time is $\frac{1}{2}(n-1)!$.
(ii) Number of circular permutations of n different things taken r at a time, when clockwise and anti-clockwise orders are taken as different, is $\frac{{}^nP_r}{r}$.
(iii) Number of circular permutations of n different things taken r at a time, when clockwise and anti-clockwise orders are not different, is $\frac{{}^nP_r}{2r}$.

SPEED BOOSTER

The number of ways in which m things of one type and n things of another type can be arranged in the form of garland, so that all the second type of things come together is $\frac{m!n!}{2}$ and no two things of second type come together is $\frac{(m-1)\,{}^{m}P_{n}}{2}$.

Exponent of Prime p in $n!$

Exponent of a prime p in $n!$ is denoted by $E_p(n!)$ and is given by

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots$$

Practice SHOTS

1. If $a_n = n\,(n!)$, then $\sum_{r=1}^{100} a_r$ is equal to

(a) $101!$ (b) $100! - 1$
(c) $101! - 1$ (d) $101! + 1$

2. If $n!, 3 \times n!$ and $(n+1)!$ are in GP, then $n!, 5 \times n!$ and $(n+1)!$ are in
(a) AP (b) GP
(c) HP (d) None of these

3. The exponent of 15 in 100!, is
(a) 48 (b) 24
(c) 25 (d) 23

4. The number of zeros at the end of 100!, is
(a) 22 (b) 23
(c) 24 (d) 25

5. The number of ways in which n distinct balls can be put into three boxes so that no two boxes remain empty, is
(a) 3^n (b) $3^n - 1$ (c) $3^n - 2$ (d) $3^n - 3$

6. A rectangle with sides of lengths $(2n-1)$ and $(2m-1)$ units is divided into squares of unit length. The number of rectangles which can be formed with sides of odd length, is
(a) m^2n^2
(b) $mn\,(m+1)\,(n+1)$
(c) 4^{m+n-1}
(d) None of the above

7. The total number of times the digit 3 will be written when listing integers from 1 to 1000 is
(a) 300 (b) 297 (c) 243 (d) 273

8. Five persons entered the lift cabin on the ground floor of an 8 floor house. Suppose each of them can leave the cabin independently at any floor beginning with the first. The total number of ways in which each of the five persons can leave the cabin at any one of the 7 floors, is
(a) 5^7 (b) 2520 (c) 7^5 (d) 35

9. Five boys and five girls make a line with the boys and girls alternating. The number of ways of making the line, is
(a) $(5!)^2$ (b) 10!
(c) $5! + 5!$ (d) $2\,(5!)^2$

10. How many 5 letter words, with or without meaning, can be formed out of the letter of the word 'EQUATION', if repetition of letters is not allowed?
(a) 126 (b) 5^9
(c) 9^5 (d) 15120

COMBINATION

Each of the different selections made by taking some or all of a number of distinct objects or items, irrespective of their arrangements or order in which they are placed, is called a combination.

Consider the first three letters of the English alphabet. All possible combination (selection) of these three letters taken two at a time are ab, bc, ca. Here, selection ab is same as ba. Similarly, selections bc and ca are same as cb and ac, respectively.

The number of combinations of n different things taken r at a time, is denoted as ${}^{n}C_r$,

where $${}^{n}C_r = \frac{n!}{(n-r)!\,r!}$$

Also, $${}^{n}C_r = \frac{{}^{n}P_r}{r!} \Rightarrow {}^{n}P_r = {}^{n}C_r \cdot r!$$

Properties of Combination

(i) If ${}^{n}C_r = {}^{n}C_p$, then either $r = p$ or $r + p = n$

(ii) ${}^{n}C_r = \frac{{}^{n}P_r}{r!}$

(iii) ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$

(iv) ${}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \ldots + {}^{n}C_n = 2^n$

(v) ${}^{n}C_0 + {}^{n}C_2 + {}^{n}C_4 + \ldots = {}^{n}C_1 + {}^{n}C_3 + \ldots = 2^{n-1}$

(vi) ${}^{n}C_r = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n}{r} \cdot \frac{n-1}{r-1} \cdot {}^{n-2}C_{r-2}$

(vii) ${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n = 2^{2n}$

(viii) ${}^{n}C_n + {}^{n+1}C_n + {}^{n+2}C_n + \ldots + {}^{2n-1}C_n = {}^{2n}C_{n+1}$

Restricted Combination

Number of combinations of n different things taken r at a time

(i) When p particular things are always included $= {}^{n-p}C_{r-p}$

(ii) When p particular things are never included $= {}^{n-p}C_{r}$

(iii) When p particular things are not together in any selection $= {}^{n}C_{r} - {}^{n-p}C_{r-p}$

SPEED BOOSTER

(i) Number of selections of r consecutive things out of n things along a circle $= \begin{cases} n, & \text{when } r < n \\ 1, & \text{when } r = n \end{cases}$

(ii) Number of selections of r consecutive things out of n things in a row $= (n - r + 1)$

Selection of One or More Objects

When we select zero or more objects from given number of objects. Then, following cases are possible

Selection from Different Objects

(i) Number of selections of zero or more things out of n different things, is ${}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \ldots + {}^{n}C_n = 2^n$

(ii) Number of selections of atleast one or more things out of n different things is ${}^{n}C_1 + {}^{n}C_2 + {}^{n}C_3 + \ldots + {}^{n}C_n = 2^n - 1$

Selection from Identical Objects

(i) The number of selections of r objects out of n identical objects is 1.

(ii) Total number of selections of zero or more objects from n identical objects is $(n + 1)$.

(iii) The total number of selections of atleast one out of $a_1 + a_2 + \ldots + a_n$ objects, where a_1 objects are alike of one kind, a_2 objects are alike of second kind, ... , a_n objects are alike of nth kind, is $(a_1 + 1)(a_2 + 1)\ldots\{(a_n + 1) - 1\}$.

Selection when Both Identical and Distinct Objects are Present

(i) The number of possible combinations of n objects of which p are identical taken r at a time, is

$$\begin{cases} {}^{n-p}C_r + {}^{n-p}C_{r-1} + {}^{n-p}C_{r-2} + \ldots + {}^{n-p}C_0, & \text{if } r \le p \\ {}^{n-p}C_r + {}^{n-p}C_{r-1} + {}^{n-p}C_{r-2} + \ldots + {}^{n-p}C_{r-p}, & \text{if } r \ge p \end{cases}$$

(ii) The number of selections taking atleast one out of $a_1 + a_2 + a_3 + \ldots + a_n + k$ objects, where a_1 are alike of one kind, a_2 are alike of second kind, ... , a_n are alike of nth kind and k are distinct objects, is $(a_1 + 1)(a_2 + 1)\ldots\{(a_n + 1)\cdot 2^k - 1\}$.

Division into Groups and Distribution

(i) Number of ways of dividing $(m + n + p)$ different objects in three groups containing m, n and p objects respectively $(m \ne n \ne p)$, is

$$\begin{cases} \dfrac{(m+n+p)!}{m!\,n!\,p!}, & \text{if order is not considered} \\ \dfrac{(m+n+p)!}{m!\,n!\,p!}\,3!, & \text{if order of groups is considered} \end{cases}$$

(ii) Number of ways of dividing $(m + n + p)$ different objects in three groups containing m, n and p objects respectively $(m = n = p)$, is

$$\begin{cases} \dfrac{(m+n+p)!}{m!\,n!\,p!\,3!} = \dfrac{(3m)!}{(m!)^3 3!} & \text{order of group is not considered} \\ \dfrac{(m+n+p)!\,3!}{m!\,n!\,p!\,3!} = \dfrac{(3m)!}{(m!)^3} & \text{order of group is considered} \end{cases}$$

(iii) Number of ways of dividing n identical objects into r groups, if blank groups are allowed, is ${}^{n+r-1}C_{r-1}$.

(iv) Number of ways of dividing n identical things into r groups, if blank groups are not allowed, is ${}^{n-1}C_{r-1}$.

(v) Number of ways of dividing n identical things into r groups such that no group contains less than m things and more than $k\,(m < k)$ things is coefficient of x^n in the expansion of $(x^m + x^{m+1} + \ldots + x^k)^r$.

(vi) The number of ways of selecting r things out of n things of which p are alike of one kind, q are alike of another kind, s are alike of third kind and so on, is coefficient of x^r in expansion of

$$[(1 + x + x^2 + \ldots + x^p)(1 + x + x^2 + \ldots + x^q)(1 + x + x^2 + \ldots + x^s)\ldots]$$

when each thing is considered atleast once = coefficient of x^{r-3} in expansion of

$$(1 + x + x^2 + \ldots + x^{p-1})(1 + x + x^2 + \ldots + x^{q-1})(1 + x + x^2 + \ldots + x^{s-1})$$

(vii) The number of ways in which n different things can be distributed into r different groups, is

$$r^n - {}^{r}C_1(r-1)^n + {}^{r}C_2(r-2)^n - \ldots + (-1)^{r-1}\,{}^{n}C_{r-1}$$

(viii) The number of ways of selecting r things out of n things of which p are alike of one kind, q are alike of second kind and $(n - p - q)$ things are all different, is coefficient of x^r in $(1 + x + x^2 + \ldots + x^p)(1 + x + x^2 + \ldots + x^q)(1 + x)^{n-p-q}$

(ix) The number of ways in which r identical things can be distributed among n persons when each person can get 0 or more things

= Coefficient of x^r in $(1 + x + x^2 + \ldots + x^r)^n$

= Coefficient of x^r in $(1 - x)^{-n} = {}^{n+r-1}C_r$ or ${}^{n+r-1}C_{n-1}$

SPEED BOOSTER

(i) The number of non-negative integral solutions of the equation $x_1 + x_2 + x_3 + \ldots + x_r = n$ is ${}^{n+r-1}C_{r-1}$.

(ii) The number of positive integral solutions of the equation $x_1 + x_2 + \ldots + x_r = n$ is ${}^{n-1}C_{r-1}$.

(iii) The number of terms in the expansion of $(a_1 + a_2 + a_3 + \ldots + a_n)^r$ is ${}^{n+r-1}C_{r-1}$.

Number of Divisors (Prime Factors)

Let N be a number which can be expressed as

$$N = p_1^{a_1} \cdot p_2^{a_2} \ldots p_k^{a_k},$$

where, $p_i, i = 1, 2, \ldots, k$ are prime numbers.

and $a_i, i = 1, 2 \ldots k$ are positive integers. Then,

(i) The total number of divisors of N including 1 and N is $(a_1 + 1)(a_2 + 1)\ldots(a_k + 1)$.

(ii) The total number of divisors excluding 1 and N is $(a_1 + 1)(a_2 + 1)\ldots(a_k + 1) - 2$.

(iii) The total number of divisors excluding 1 or N is $(a_1 + 1)(a_2 + 1)(a_3 + 1)\ldots(a_k + 1) - 1$.

(iv) Sum of the distinct positive integral divisors of N is

$$\left(\frac{p_1^{a_1+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{a_2+1} - 1}{p_2 - 1}\right)\left(\frac{p_3^{a_3+1} - 1}{p_3 - 1}\right)\ldots\left(\frac{p_k^{a_k+1} - 1}{p_k - 1}\right).$$

(v) The number of ways in which N can be resolved as a product of two factors is

$$\begin{cases} \frac{1}{2}(a_1 + 1)(a_2 + 1)\ldots(a_k + 1), \text{ if } N \text{ is not a perfect square} \\ \frac{1}{2}(a_1 + 1)(a_2 + 1)\ldots(a_k + 1) + 1, \text{ if } N \text{ is a perfect square} \end{cases}$$

(vi) The number of ways in which a composite number N can be resolved into two factors which are relatively prime to each other is equal to 2^{n-1}, where n is the number of different factors in N.

Some Results Based on Geometrical Problems

(i) If n distinct points are given in the plane such that no three of which are collinear, then the number of line segments formed is ${}^nC_2$ and if m of these points are collinear $(m \geq 3)$, then number of line segments is $({}^nC_2 - {}^mC_2 + 1)$.

(ii) The number of diagonals in an n sided closed polygon $= {}^nC_2 - n$.

(iii) If n distinct points are given in the plane such that no three of which are collinear, then number of triangles formed $= {}^nC_3$

If m of these points are collinear $(m \geq 3)$, then number of triangles formed $= {}^nC_3 - {}^mC_3$

(iv) If n distinct points are given on the circumference of circle, then

(a) number of straight lines $= {}^nC_2$

(b) number of triangles $= {}^nC_3$

(c) number of quadrilaterals $= {}^nC_4$

(d) number of k sided polygons $= {}^nC_k$

(v) In a square of size $n \times n$, number of rectangles of any size is equal to $\sum_{r=1}^{n} r^3$ and number of squares of any size is $\sum_{r=1}^{n} r^2$.

(vi) Number of rectangles of any size in a rectangle of size $n \times p\,(n < p)$ is $\frac{np}{4}(n + 1)(p + 1)$ and number of squares of any size is $\sum_{r=1}^{n}(n + 1 - r)(p + 1 - r)$.

(vii) If m parallel vertical lines are intersected by n parallel horizontal lines, then number of parallelograms, so formed is ${}^mC_2 \times {}^nC_2 = \frac{mn(m-1)(n-1)}{4}$.

Derangement

Any change in the given order of things is called a derangement.

(i) If n things form an arrangement in a row, then the number of ways in which they can be derange, so that none of them occupies its original place, is D_n,

where $D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right)$

(ii) $D_n =$ Number of ways, if all n things goes to wrong place. Thus, $D_r =$ Number of ways, if r things goes to wrong place. If r goes to wrong places out of n, then $(n - r)$ goes to correct places $= {}^nC_{n-r}D_r$

where, $D_r = r!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^r \frac{1}{r!}\right]$

Practice SHOTS

11. The number of positive integers satisfying the inequality ${}^{n+1}C_{n-2} - {}^{n+1}C_{n-1} \leq 100$, is

(a) 9 (b) 8

(c) 5 (d) None of these

12. If ${}^{n-1}C_r = (k^2 - 3)\,{}^nC_{r+1}$, then k belongs to the interval

(a) $[-\sqrt{3}, \sqrt{3}]$ (b) $(-\infty, -2)$

(c) $(2, \infty)$ (d) $(\sqrt{3}, 2]$

13. If $n \geq r$, then ${}^rC_r + {}^{r+1}C_r + {}^{r+2}C_r + \ldots + {}^nC_r$ is equal to

(a) ${}^{n+1}C_r$ (b) ${}^{n+1}C_{r+1}$

(c) ${}^{n+2}C_{r+1}$ (d) ${}^{n+2}C_{r+2}$

14. From 12 books, the difference between the number of ways of selection of 5 books when one specified book is always excluded and one specific book is always included is

(a) 64 (b) 118

(c) 132 (d) 330

15. From 6 different novels and three different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on the self, so that the dictionary is always in the middle. Then, the number of such arrangements is

(a) atleast 500 but less than 750
(b) atleast 750 but less than 1000
(c) atleast 1000
(d) less than 500

16. Six 'x' have to be placed in the squares of the figure given, such that each row contains atleast one x, this can be done in

(a) 24 ways (b) 28 ways (c) 26 ways (d) 36 ways

17. The number of odd proper divisors and even proper divisors of 5040 are respectively

(a) 40, 10 (b) 11, 47
(c) 12, 39 (d) None of these

18. Let T_n denotes the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, then n equals

(a) 5 (b) 7 (c) 6 (d) 4

19. A father with 8 children takes 3 at a time to the zoological garden, as often as he can without taking the same 3 children together more than once. The number of times he will go to the garden, is

(a) 112 (b) 56
(c) 336 (d) None of these

20. Everybody in a room shakes hands with everybody else. If total number of hand shakes is 66, then total number of persons in the room is

(a) 9 (b) 12 (c) 10 (d) 14

Answers

1. (c)	**2.** (a)	**3.** (b)	**4.** (c)	**5.** (d)
6. (a)	**7.** (a)	**8.** (c)	**9.** (d)	**10.** (d)
11. (b)	**12.** (d)	**13.** (b)	**14.** (c)	**15.** (c)
16. (c)	**17.** (b)	**18.** (b)	**19.** (b)	**20.** (b)

MASTER STROKES

21. The value of $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n!$ is

(a) $(n+1)!$ (b) $(n+1)! + 1$
(c) $(n+1)! - 1$ (d) None of these

22. Sum of the series $\sum_{r=1}^{n} (r^2 + 1)\, r!$ is

(a) $(n+1)!$ (b) $(n+2)! - 1$
(c) $n(n+1)!$ (d) None of these

23. The exponent of 12 in 100! is

(a) 24 (b) 46 (c) 47 (d) 48

24. For $2 \le r \le n$, ${}^nC_r + 2\,{}^nC_{r-1} + {}^nC_{r-2}$ equals

(a) ${}^{n+1}C_{r-1}$ (b) $2\,{}^{n+1}C_{r+1}$ (c) ${}^{n+2}C_r$ (d) $2\,{}^{n+2}C_r$

25. $\sum_{k=m}^{n} {}^kC_r$ is equal to

(a) ${}^{n+1}C_{r+1}$ (b) ${}^{n+1}C_{r+1} - {}^mC_r$
(c) ${}^{n+1}C_{r+1} + {}^mC_{r+1}$ (d) ${}^{n+1}C_{r+1} - {}^mC_{r+1}$

26. Ten different letters of an alphabet are given. Words with five letters are formed from these given letters. The number of words which have atleast one letter repeated is

(a) 10^5 (b) 30240 (c) 69760 (d) 86490

27. The sum of all the numbers that can be formed with the digits 2, 3, 4, 5 taken all at a time, is

(a) $(2+3+4+5) \times 3! \times \dfrac{10^4 - 1}{10 - 1}$

(b) $(2+3+4+5) \times \dfrac{10^4 - 1}{10 - 1}$

(c) $3! \times \dfrac{10^4 - 1}{10 - 1}$

(d) $(2+3+4+5) \times 3!$

28. On how many nights may a watch of 4 men be arranged by a security agency from a crew of 16 so that no two watches are identical? On how many of these nights would a particular person be off-duty?

(a) 1800, 1600 (b) 1820, 1365
(c) 1840, 1340 (d) 1860, 1640

29. There are m points on one straight line AB and n points on another straight line AC, none of them being A. How many triangles can be formed with these points as vertices, if point A is also included?

(a) $\dfrac{mn}{2}(m+n-2)$ (b) $\dfrac{mn}{2}(m+n-1)$
(c) $\dfrac{mn}{2}(m+n)$ (d) $\dfrac{mn}{2}(m+n+1)$

30. The number of all permutations of n different objects taken r at a time, when a particular objects is to be always included in each arrangements, is

(a) ${}^nC_r \times r!$
(b) ${}^{n-1}C_{r-1} \times r!$
(c) ${}^{n-1}C_{r-1} \times (r-1)!$
(d) ${}^{n-1}C_r \times r!$

31. m men and n women are to be seated in a row so that no two women sit together. If $m > n$, then the number of ways in which they can be seated is

(a) $\dfrac{m!\,n!}{(m+n)!}$ (b) $\dfrac{m!\,(m+1)!}{(m-n+1)!}$
(c) $\dfrac{m!\,(m+1)!}{(m+n+1)!}$ (d) $\dfrac{m!\,(m+1)!}{(m+n)!}$

32. The total number of natural numbers lying between 20000 and 60000, the sum of whose digit is even, is

(a) 40000 (b) 20000
(c) 5000 (d) 25000

33. At an election, a voter may vote for any number of candidates, not greater than the number to be elected. There are 10 candidates and 4 are to be elected. If a voter votes for atleast one candidate, then the number of ways in which he can vote, is
(a) 5040 *(b)* 6210 *(c)* 385 *(d)* 1110

34. The set $S = \{1, 2, 3, \ldots, 12\}$ is to be partitioned into three sets A, B, C of equal size. Thus, $A \cup B \cup C = S$, $A \cap B = B \cap C = A \cap C = \phi$. The number of ways to partition S is
(a) $\frac{12!}{(4!)^3}$ *(b)* $\frac{12!}{(3!)^4}$ *(c)* $\frac{12!}{3!(4!)^3}$ *(d)* $\frac{12!}{3!(3!)^4}$

35. The number of ways in which four different letters can be put in the correspondingly five addressed envelopes so that no letters is put in the correct envelope, is
(a) 6 *(b)* 7 *(c)* 8 *(d)* 9

36. There are 20 persons among whom two are brothers. The number of ways in which we can arrange them around a circle so that there is exactly one person between the two brothers, is
(a) 18! *(b)* 17! × 2! *(c)* 18! × 2! *(d)* 20!

37. The number of factors (excluding 1 and the expression itself) of the product of $a^7b^4c^3def$, where a, b, c, d, e, f are all prime numbers, is
(a) 1279 *(b)* 1278 *(c)* 1280 *(d)* 1277

38. The number of divisors of 10800 which are divisible by 15, is
(a) 10 *(b)* 20 *(c)* 15 *(d)* 30

39. In how many ways can a pack of 52 cards be divided into 4 sets, three of them having 17 cards each and the fourth just one card?
(a) $\frac{52!}{(17!)^3\ 3!}$ *(b)* $\frac{52!}{(17!)^3}$ *(c)* $\frac{52!}{51!(17!)^3}$ *(d)* $\frac{52!}{51!}$

40. In how many ways can 20 identical toys be distributed among 4 children so that each one gets atleast 3 toys?
(a) ${}^{11}C_2$ *(b)* ${}^{12}C_3$ *(c)* ${}^{11}C_3$ *(d)* ${}^{12}C_4$

41. The total number of integral solutions of the equation $x + y + z + t = 29$, where $x \geq 1$, $y \geq 2$, $z \geq 3$ and $t \geq 0$, is
(a) ${}^{26}C_4$ *(b)* ${}^{26}C_3$ *(c)* ${}^{22}C_3$ *(d)* ${}^{29}C_3$

42. In how many ways can four persons, each throwing a dice once, make a sum of 13?
(a) 220 *(b)* 180 *(c)* 140 *(d)* 80

43. Ravish writes letters to his five friends and addresses the corresponding envelopes. The total number of ways in which letters can be placed in the envelopes so that atleast two of them are in the wrong envelopes, is
(a) 120 *(b)* 118 *(c)* 119 *(d)* 121

44. If $n = {}^mC_2$, then the value of ${}^nC_2$ is
(a) ${}^{m+1}C_4$ *(b)* ${}^{m-1}C_4$ *(c)* ${}^{m+2}C_4$ *(d)* $3\ {}^{m+3}C_4$

45. The number of different matrices that can be formed with elements 0, 1, 2 or 3 each matrix having 4 elements, is
(a) 3×2^4 *(b)* 2×4^4 *(c)* 3×4^4 *(d)* 2×2^4

46. The total number of 5-digit numbers of different digits in which the digit in the middle is the largest, is
(a) 30 × 3! *(b)* 33 × 3!
(c) $\sum_{n=4}^{9} {}^nC_4 \times 4!$ *(d)* None of these

47. Six boys and six girls sit along a line alternatively in x ways and along a circle (again alternatively in y ways), then
(a) $x = y$ *(b)* $y = 12x$
(c) $x = 10y$ *(d)* $x = 12y$

48. In a certain test, there are n questions. In this test, 2^{n-k} students gave wrong answers to atleast k questions, where $k = 1, 2, 3, \ldots, n$. If the total number of wrong answers is 2047, then n is equal to
(a) 10 *(b)* 11 *(c)* 12 *(d)* 13

49. Given that n is odd, the number of ways in which three numbers in AP can be selected from 1, 2, 3, ..., n is
(a) $\frac{(n-1)^2}{2}$ *(b)* $\frac{(n+1)^2}{4}$ *(c)* $\frac{(n+1)^2}{2}$ *(d)* $\frac{(n-1)^2}{4}$

50. If one quarter of all three elements subsets of the set $A = \{a_1, a_2, a_3, \ldots, a_n\}$ contains the element a_3, then n equals
(a) 10 *(b)* 12 *(c)* 14 *(d)* 16

51. Consider all possible permutations of the letter of the word ENDEANOEL. The number of permutations containing the word ENDEA, is
(a) 5! *(b)* 2 × 5! *(c)* 7 × 5! *(d)* 21 × 5!

52. If ${}^mC_r = 0$ for $r > m$, then the sum $\sum_{r=0}^{m} {}^{18}C_r\ {}^{20}C_{m-r}$ is maximum when m equals
(a) 20 *(b)* 19 *(c)* 10 *(d)* 9

53. There are two urns. Urn A has 3 distinct red balls and Urn B has 9 distinct blue balls. From each urn, two balls are taken out at random and then transferred to the other. The number of ways in which this can be done is
(a) 66 *(b)* 108 *(c)* 3 *(d)* 36

54. Suppose that six students, including Madhu and Puja are having six beds arranged in a row. Further, suppose that Madhu does not want a bed adjacent to Puja. Then, the number of ways, the beds can be allotted to students is
(a) 384 *(b)* 264 *(c)* 480 *(d)* 600

55. The number of arrangements that can be formed from the letters a, b, c, d, e, f taken 3 at a time without repetition and each arrangement containing atleast one vowel, is
(a) 128 *(b)* 24 *(c)* 72 *(d)* 96

56. The number of points, having both coordinates as integers, that lie in the interior of the triangle with vertices (0, 0), (0, 41) and (41, 0), is
(a) 820 *(b)* 780 *(c)* 901 *(d)* 861

57. There are 10 girls and 8 boys in a classroom including Mr Ravi, Ms Rani and Ms Radha. A list of speakers consisting of 8 girls and 6 boys has to be prepared. Mr Ravi refuses to speak, if Ms Rani is a speaker. Ms Rani refuses to speak, if Ms Radha is a speaker. The number of ways the list can be prepared is
(a) 202 *(b)* 308 *(c)* 567 *(d)* 952

58. There are 10 points in a plane, out of these 6 are collinear. If N is the number of triangles formed by joining these points, then
(a) $100 < N \leq 140$ *(b)* $140 < N \leq 190$
(c) $N > 190$ *(d)* $N \leq 100$

59. The number of integers greater than 6000 that can be formed, using the digits 3, 5, 6, 7 and 8 without repetition is
(a) 120 *(b)* 72 *(c)* 216 *(d)* 192

60. The total number of ways in which 5 balls of different colours can be distributed among 3 persons so that each person gets atleast one ball is
(a) 75 *(b)* 150 *(c)* 210 *(d)* 243

61. Let n be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that all the girls stand consecutively in the queue. Let m be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that exactly four girls stand consecutively in the queue. Then, the value of $\frac{m}{n}$ is
(a) 4 *(b)* 5 *(c)* 6 *(d)* 7

Answers

21. (c)	**22.** (c)	**23.** (c)	**24.** (c)	**25.** (d)
26. (c)	**27.** (a)	**28.** (b)	**29.** (c)	**30.** (b)
31. (b)	**32.** (b)	**33.** (c)	**34.** (a)	**35.** (d)
36. (c)	**37.** (b)	**38.** (d)	**39.** (a)	**40.** (c)
41. (b)	**42.** (c)	**43.** (c)	**44.** (d)	**45.** (c)
46. (d)	**47.** (d)	**48.** (b)	**49.** (d)	**50.** (b)
51. (a)	**52.** (b)	**53.** (b)	**54.** (c)	**55.** (d)
56. (b)	**57.** (b)	**58.** (d)	**59.** (d)	**60.** (b)
61. (b)				

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BINOMIAL THEOREM

Binomial theorem is used to expand the expressions like $(x+a)^n$, $(x-a)^n$, $(1+x)^n$, etc.

Binomial Theorem

If n is a positive integer, then the expansion of $(x+a)^n$ can be written as

$$(x+a)^n = {}^nC_0 x^n a^0 + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \ldots + {}^nC_r x^{n-r} a^r + \ldots + {}^nC_n x^0 a^n$$

Observations

(i) This expansion has $(n+1)$ terms.

(ii) In each term, sum of powers of 'x' and 'a' is always equal to n.

(iii) ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_r, \ldots, {}^nC_n$ are called binomial coefficients.

(iv) Binomial coefficient of the terms equidistant from beginning and end are same, i.e. ${}^nC_r = {}^nC_{n-r}$.

(v) pth term from the end is $(n-p+2)$th term from the beginning.

(vi) $S = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, a^r = \sum_{r=0}^{n} {}^nC_{n-r}\, x^r\, a^{n-r}$

(vii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(viii) $r \cdot {}^nC_r = n \cdot {}^{n-1}C_{r-1}$

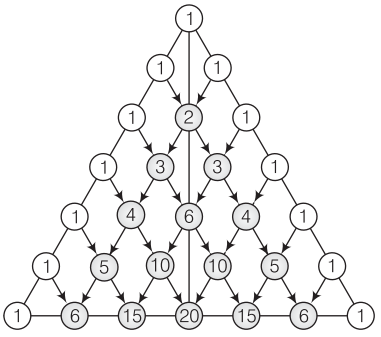
(ix) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$

(x) If ${}^nC_x = {}^nC_y \Rightarrow$ either $x = y$ or $x + y = n$

So, ${}^nC_r = {}^nC_{n-r} = \dfrac{n!}{r!\,n-r!}$

Pascal's Triangle

The coefficients of binomial expansion follow a particular pattern represented below



In the above triangle, any entry except the first and the last in the row is the sum of two entries above it in the preceding row. The above pattern is called Pascal's triangle.

Some Standard Expansions

(i) $(x-a)^n = {}^nC_0 x^n a^0 - {}^nC_1 x^{n-1} a^1 + {}^nC_2 x^{n-2} a^2 - {}^nC_3 x^{n-3} a^3 + \ldots + (-1)^n {}^nC_n x^0 a^n$

i.e. $(x-a)^n = \sum_{r=0}^{n} (-1)^r {}^nC_r\, x^{n-r} a^r$

(ii) $(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$,

i.e. $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$

(iii) $(1-x)^n = {}^nC_0 x^0 - {}^nC_1 x + {}^nC_2 x^2 - \ldots + (-1)^n {}^nC_n x^n$

i.e. $(1-x)^n = \sum_{r=0}^{n} (-1)^r {}^nC_r x^r$

(iv) $(x+a)^n + (x-a)^n = 2[{}^nC_0 x^n a^0 + {}^nC_2 x^{n-2} a^2 + \ldots]$

(v) $(x+a)^n - (x-a)^n = 2[{}^nC_1 x^{n-1} a + {}^nC_3 x^{n-3} a^3 + \ldots]$

Tips to Find Coefficient of Required Terms

- Coefficient of x^r in expansion of $(1+x)^n$ is ${}^nC_r$.
- Coefficient of x^r in expansion of $(1+ax)^n$ is ${}^nC_r a^r$.
- Coefficient of x^r in expansion of $(a+bx)^n$ is ${}^nC_r b^r a^{n-r}$.
- Coefficient of x^r in expansion of $(1+x^p)^n$ is ${}^nC_{r/p}$, if r is a multiple of p.

General Term

In the expansion of $(x+a)^n$, general term is $(r+1)$th term which can be represented as T_{r+1}.

$\therefore T_{r+1} = {}^nC_r\, x^{n-r}\, a^r = {}^nC_r$ (First term)$^{n-r}$ (Second term)^r

Note Independent term or constant term of binomial expansion is the term in which exponent of variable is zero.

To find the term independent of x in the expansion of $(x+a)^n$, *we use following steps.*

***Step* I** Write down the general term in the expansion of $(x+a)^n$

i.e. $(r+1)$th term is $T_{r+1} = {}^nC_r\, x^{n-r} a^r$.

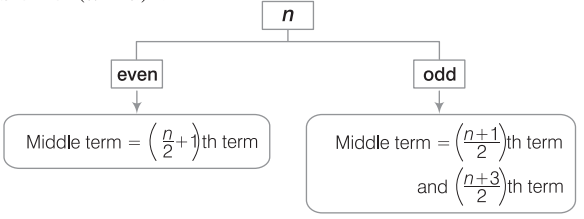
***Step* II** Separate constant and variable by grouping.

***Step* III** Put the index of x obtained in step II equal to zero and simplify for 'r'.

We will get the value of r for which there exists a term independent of x in the expansion.

Middle Term

The middle term depends on the value of n (even or odd). The following diagram will help you to find the middle term in the expansion of $(x+a)^n$.



Note (i) If there are two middle terms in the expansion, then their binomial coefficients are equal.

(ii) Binomial coefficient of middle term is the greatest binomial coefficient.

Greatest Term

To find the greatest term in the expansion of $(1+x)^n$, follow the steps given below.

***Step* I** Find the value of $\frac{x(n+1)}{x+1}$. Let $\frac{x(n+1)}{x+1} = m$

***Step* II** Then, there will be two cases arise.

***Case* I** If m is integer, then T_m or T_{m+1} are numerically equal and both are greatest term.

***Case* II** If m is not integer, then $T_{[m]+1}$ is greatest term, where $[\cdot]$ denotes the greatest integer function.

Method to Find Remainder Using Binomial Theorem

To find the remainder, when a^n is divided by b, we adjust the power of a to a^k, which is very close to b or multiple of b, if possible with difference 1. Also, it is to be noted down the remainder is always positive.

e.g. When 5^{99} is divided by 13, then

$$N = 5^{99} = 5(5^{98}) = 5(5^2)^{49} = 5(26-1)^{49}$$

$$\Rightarrow \quad N = 5[{}^{49}C_0 26^{49} - {}^{49}C_1 26^{48} \ldots {}^{49}C_{48}26 - {}^{49}C_{49}]$$

$$\Rightarrow \quad N = 5[{}^{49}C_0 26^{49} - {}^{49}C_1 26^{48} + \ldots + {}^{49}C_{48}26] - 5$$

$$\Rightarrow \quad \frac{N}{13} = \frac{5}{13}[{}^{49}C_0 26^{49} - {}^{49}C_1 26^{48} + \ldots + {}^{49}C_{48}26] - \frac{5}{13}$$

$$\Rightarrow \quad \frac{N}{13} = \frac{5}{13}[{}^{49}C_0 26^{49} - {}^{49}C_1 26^{48} + \ldots + {}^{49}C_{48}26] - 1 + \frac{8}{13}$$

[here, 1 is added or subtracted to make remainder positive]

$\therefore$ Remainder = 8

Properties of Binomial Coefficient

Following are important properties of binomial coefficients. These results will help you in great extent to minimise the problem solving time.

(i) If n is even, then greatest coefficient is ${}^nC_{n/2}$.

(ii) If n is odd, then greatest coefficient is ${}^nC_{\frac{n-1}{2}}$ or ${}^nC_{\frac{n+1}{2}}$.

(iii) $C_0 + C_1 + C_2 + \ldots + C_n = 2^n$

(iv) $C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^n C_n = 0$

(v) $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$

(vi) $C_0^2 + C_2^2 + C_4^2 + \ldots + C_n^2 = \frac{(2n)!}{n!\,n!} = {}^{2n}C_n$

(vii) $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2} \cdot {}^nC_{n/2}, & \text{if } n \text{ is even} \end{cases}$

(viii) $C_0C_1 + C_1C_2 + C_2C_3 + \ldots + C_{n-1}C_n = {}^{2n}C_{n+1}$

(ix) $C_0C_r + C_1C_{r+1} + \ldots + C_{n-r}C_n = {}^{2n}C_{n-r}$ or ${}^{2n}C_{n+r}$

(x) $1C_1 + 2C_2 + 3C_3 + \ldots + nC_n = n \cdot 2^{n-1}$

(xi) $C_1 - 2C_2 + 3C_3 - \ldots = 0$

(xii) $C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n = (n+2)2^{n-1}$

(xiii) ${}^nC_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + \ldots + {}^{n+n}C_n = {}^{2n+1}C_{n+1}$

(xiv) ${}^nC_0\,{}^{2n}C_n - {}^nC_1\,{}^{2n-1}C_n + {}^nC_2\,{}^{2n-2}C_n + \ldots + (-1)^n\,{}^nC_n\,{}^nC_n = 1$

Important Theorem

If $(\sqrt{A}+B)^n = I + f$, where I and n are positive integers, n being odd and $0 \le f < 1$, then $(I+f)\cdot f = k^n$, where $A - B^2 = k > 0$ and $\sqrt{A} - B < 1$. If n is an even integer, then $(\sqrt{A}+B)^n + (\sqrt{A}-B)^n = I + f + f'$. Hence, LHS and I are integers. Then, $f + f'$ is also an integer $\Rightarrow f + f' = 1$

$\therefore \quad f'(1-f) \Rightarrow (\sqrt{A}+B)^n(\sqrt{A}-B)^n = (A-B^2)^n = k^n$

Sum of the Series, when *i* and *j* are Dependent

If sum of the series $\sum\limits_{0\le i<j\le n} f(i)f(j)$ is given, then i and j are not independent. In this summation, three types of terms occur, those when $i < j$, $i > j$ and $i = j$. Also, sum of terms, when $i < j$ is equal to sum of the terms when $i > j$, if $f(i)$ and $f(j)$ are symmetrical.

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)f(j) = \sum\sum_{0\le i<j\le n} f(i)f(j) + \sum\sum_{0\le i<j\le n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)f(j) = 2\sum\sum_{0\le i<j\le n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$\text{Hence, } \sum\sum_{0\le i<j\le n} f(i)f(j) = \frac{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} f(i)f(j) - \sum\sum\limits_{i=j} f(i)f(j)}{2}$$

$$\text{Also, } \sum\sum_{0\le i<j\le n} f(i)f(j) = \frac{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} f(i)f(j) + \sum\sum\limits_{i=j} f(i)f(j)}{2}$$

$[\because 0 \le i \le j \le n]$

Some Important Results

(i) $\sum\sum_{0\le i<j\le n} {}^nC_i \cdot {}^nC_j = \frac{2^{2n} - {}^{2n}C_n}{2}$

(ii) $\sum\sum_{0\le i\le j\le n} {}^nC_i \cdot {}^nC_j = \frac{2^{2n} + {}^{2n}C_n}{2}$

(iii) $\sum\sum_{i\ne j} {}^nC_i \cdot {}^nC_j = 2^{2n} - {}^{2n}C_n$

Note (i) Here, $|x| < 1$ is necessary, if n is rational number other than whole number.

(ii) There are infinite number of terms in the expansion of $(1+x)^n$, when n is integer or fraction.

(iii) Make the first term unity, if it is other than unity, when n is integer or fractional.

e.g. $(a+bx)^n = a^n\left(1 + \frac{bx}{a}\right)^n$, here $\left|\frac{bx}{a}\right| < 1$

(iv) The general term in the expansion of $(1+x)^n$ is given by $T_{r+1} = \frac{n(n-1)(n-2)\ldots\{n-(r-1)\}}{r!}x^r$

(v) Expansion of

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r + \ldots$$

Thus in the expansion of $(1-x)^{-n}$, coefficient of x^r is ${}^{n+r-1}C_r$.

Multinomial Theorem

The expansion of $(x_1 + x_2 + x_3 + \ldots + x_r)^n$

$$= \Sigma \frac{n!}{n_1!\, n_2!\, n_3! \ldots n_r!} x_1^{n_1} \cdot x_2^{n_2} \ldots x_r^{n_r},$$

where $n_1 + n_2 + n_3 + \ldots + n_r = n$

and $0 \le n_1, n_2, \ldots, n_r (\le n)$

Some Important Points

(i) Coefficient of $x_1^{n_1}, x_2^{n_2}, \ldots, x_r^{n_r}$ in expansion of $(x_1 + x_2 + x_3 + \ldots + x_r)^n$ is $\dfrac{n!}{n_1!\, n_2!\, n_3! \ldots n_r!}$.

(ii) The greatest coefficient in expansion of $(x_1 + x_2 + x_3 + \ldots + x_m)^n$ *is* $\dfrac{n!}{(q!)^{m-r}[(q+1)!]^r}$.

where, q is quotient and r is remainder, when n is divided by m.

(iii) If n is positive integer and $a_1, a_2, \ldots, a_m \in C$, then coefficient of x^r in the expansion of $(a_1 + a_2 x + a_3 x^2 + \ldots + a_{n-1} x^{m-1})^n$ is

$$\Sigma \frac{n!}{n_1!\, n_2!\, n_3! \ldots n_m!} \cdot a_1^{n_1} \cdot a_2^{n_2} \ldots, a_m^{n_m}$$

Here, $n_1 + n_2 + \ldots + n_m = n$

and $n_2 + 2n_3 + 3n_4 + \ldots + (m-1)n_m = r$

(iv) The number of dissimilar or distinct term in the multinomial expansion of $(a_1 + a_2 + \ldots + a_m)^n$ is ${}^{n+m-1}C_{m-1}$.

Observe the example to find coefficient of x^n of multinomial expansion.

Find the coefficient of x^7 in expansion of $(1 + 3x - 2x^3)^{10}$.

Sol. Coefficient of x^7 in the expansion of $(1 + 3x - 2x^3)^{10}$

$$= \Sigma \frac{10!}{n_1!\, n_2!\, n_3!} (1)^{n_1} (3x)^{n_2} (-2x^3)^{n_3}$$

where, $n_1 + n_2 + n_3 = 10$

and $n_2 + 3n_3 = 7$

	$n_2 + 3n_3 = 7$		
n_1	n_2	n_3	$n_1 + n_2 + n_3$
3	7	0	10
5	4	1	10
7	1	2	10

$\therefore$ Coefficient of x^7

$$= \frac{10!}{3!7!0!}(1)^3(3)^7(-2)^0 + \frac{10!}{5!4!1!}(1)^5(3)^4(-2)^1 + \frac{10!}{7!1!2!}(1)^7(3)^1(-2)^2$$

$$= 262440 - 204120 + 4320 = 62640$$

Binomial Theorem for Any Index

Let n be a rational number and x be a real number, such that $|x| < 1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots \infty$$

Note (i) The condition $|x| < 1$ is unnecessary, if n is a whole number while the same condition is essential, if n is a rational number other than a whole number.

(ii) There are infinite number of terms in the expansion of $(1+x)^n$, when n is a negative integer or a fraction.

(iii) If n is a positive integer, then the above expansion contains $(n+1)$ terms and coincide with

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$

Expansion of $(x + a)^n$

To expand $(x + a)^n$, where n is the either a negative integer or a fraction, *we have following two cases*

***Case* I** When $x > a$, then $\dfrac{a}{x} < 1$.

In this case, we have $(x+a)^n = \left\{x\left(1 + \dfrac{a}{x}\right)\right\}^n$

$$= x^n \left(1 + \frac{a}{x}\right)^n = x^n \left\{1 + n \cdot \frac{a}{x} + \frac{n(n-1)}{2!} \cdot \left(\frac{a}{x}\right)^2 + \ldots\right\}$$

***Case* II** When $x < a$, then $\dfrac{x}{a} < 1$.

In this case, we have

$$(x+a)^n = \left\{a\left(1 + \frac{x}{a}\right)\right\}^n = a^n \left(1 + \frac{x}{a}\right)^n$$

$$= a^n \left\{1 + n \cdot \frac{x}{a} + \frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{x}{a}\right)^3 + \ldots\right\}$$

General Term

The general term in the expansion of $(1+x)^n$ is given by

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$$

SOME STANDARD EXPANSIONS

(i) $(1+x)^{-n} = 1 - nx + \dfrac{n(n+1)}{2!}x^2 - \dfrac{n(n+1)(n+2)}{3!}x^3 + \ldots + (-1)^r \dfrac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r + \ldots$

(ii) $(1-x)^{-n} = 1 + nx + \dfrac{n(n+1)}{2!}x^2 + \dfrac{n(n+1)(n+2)}{3!}x^3 + \ldots + \dfrac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r + \ldots$

(iii) $(1-x)^n = 1 - nx + \dfrac{n(n-1)}{2!}x^2 - \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots + (-1)^r \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots$

Exponential Series

Following are important results or observations in exponential series

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty$, where x may be any real or complex and $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$;

i.e. $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots \infty$

(ii) $a^x = e^{\log_e a^x} = e^{x \log_e a}$

$\Rightarrow a^x = 1 + \frac{x \log_e a}{1!} + \frac{(x \log_e a)^2}{2!} + \frac{(x \log_e a)^3}{3!} + \ldots \infty$

(iii) (a) $e + e^{-1} = 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots \infty\right)$

(b) $e - e^{-1} = 2\left(1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \ldots \infty\right)$

Logarithmic Series

Some important results based on logarithmic series are as follow

(i) $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \infty$

where, $-1 \le x \le 1$

(ii) $\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots \infty$

(iii) $\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \infty\right), |x| < 1$

SPEED BOOSTER

(i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \infty = \ln 2$

(ii) $\sum_{n=1}^{\infty} \frac{n}{n!} = e$ (iii) $\sum_{n=1}^{\infty} \frac{n^2}{n!} = 2e$

(iv) $\sum_{n=1}^{\infty} \frac{n^3}{n!} = 5e$ (v) $\sum_{n=1}^{\infty} \frac{n^4}{n!} = 15e$

MASTER STROKES

1. If in the expansion of $(1 + x)^m (1 - x)^n$, the coefficients of x and x^2 are 3 and -6 respectively, then m is

(a) 6 (b) 9 (c) 12 (d) 24

2. The coefficient of x^{20} in $(1 + 3x + 3x^2 + x^3)^{20}$, is

(a) ${}^{60}C_{40}$ (b) ${}^{30}C_{20}$

(c) ${}^{15}C_2$ (d) None of these

3. If O is the sum of odd terms and E that of even terms in the expansion of $(x + a)^n$, then $O^2 - E^2$ is equal to

(a) $(x^2 + a^2)^n$ (b) $(x^2 - a^2)^n$

(c) $(x - a)^{2n}$ (d) None of these

4. The coefficient of the term independent of x in the expansion of $(1 + x + 2x^3)\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$, is

(a) $\frac{1}{3}$ (b) $\frac{19}{54}$ (c) $\frac{17}{54}$ (d) $\frac{1}{4}$

5. The third term in the expansion of $\left(\frac{1}{x} + x^{\log_{10} x}\right)^5$, $x > 1$ is 1000, then x is equal to

(a) 100 (b) 10

(c) 1 (d) $\frac{1}{\sqrt{10}}$

6. The coefficient of x^6 $(0 \le r \le (n-1))$ in the expansion of $(x + 2)^{n-1} + (x + 2)^{n-2}(x + 1) + (x + 2)^{n-3}(x + 1)^2 + \ldots + (x + 1)^{n-1}$, is

(a) ${}^nC_r 2^{n-r}$ (b) ${}^nC_r(2^{n-r} - 1)$

(c) ${}^nC_r(2^{n-r} + 1)$ (d) ${}^nC_r(2^r - 1)$

7. If the second term in the expansion of $\left(\sqrt[13]{a} + \frac{a}{\sqrt{a^{-1}}}\right)^n$ is $14a^{5/2}$, then the value of $\frac{{}^nC_3}{{}^nC_2}$, is

(a) 4 (b) 3 (c) 12 (d) 6

8. If $R = (6\sqrt{6} + 14)^{2n+1}$ and $f = R - [R]$, where $[\,]$ denotes the greatest integer function, then Rf equals

(a) 20^n (b) 20^{2n} (c) 20^{2n+1} (d) 20^{n+1}

9. The greatest term in the expansion of $(1 + 3x)^{54}$ when $x = \frac{1}{3}$, is

(a) 28th (b) 25th (c) 26th (d) 24th

10. The sum $\sum_{r=0}^{m} {}^{10}C_r \times {}^{20}C_{m-r}$ is maximum, when m equals

(a) 5 (b) 10 (c) 15 (d) 20

11. The interval in which x must lie so that the greatest term in the expansion of $(1 + x)^{2n}$ has the greatest coefficient, is

(a) $\left(\frac{n-1}{n}, \frac{n}{n-1}\right)$ (b) $\left(\frac{n}{n+1}, \frac{n+1}{n}\right)$

(c) $\left(\frac{n}{n+2}, \frac{n+2}{n}\right)$ (d) None of these

12. The total number of terms in the expansion of $(2x - y + 4z)^{12}$, is

(a) 90 (b) 91

(c) 13 (d) None of these

13. The number of terms containing x^5 in the expansion of $(2 - x + 3x^2)^6$ is
(a) 3 (b) 4 (c) 5 (d) 6

14. If $C_0, C_1, C_2, \ldots, C_n$ denote the coefficients in the binomial expansion of $(1 + x)^n$, then $C_0 + 2 \cdot C_1 + 3 \cdot C_2 + \ldots + (n + 1) \cdot C_n$ is equal to
(a) $n\ 2^{n-1}$ (b) $(n + 1)\ 2^{n-1}$
(c) $(n + 2)\ 2^{n-1}$ (d) $(n + 2)\ 2^n$

15. If $C_0, C_1, C_2, \ldots, C_n$ denote the binomial coefficients in the expansion of $(1 + x)^n$, then $1^3 \cdot C_1 + 2^3 \cdot C_2 + 3^3 \cdot C_3 + \ldots + n^3 \cdot C_n$ is equal to
(a) $(n + 2)\ (n + 3)\ 2^{n-3}$ (b) $n^2(n + 3)\ 2^{n-3}$
(c) $n^2(n + 3)\ 2^n$ (d) $n(n + 1)\ (n + 2)\ 2^{n-3}$

16. If ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ denote the binomial coefficients in the expansion of $(1 + x)^n$ and $p + q = 1$, then $\sum_{r=0}^{n} r^2\ {}^nC_r\ p^r q^{n-r}$ is equal to
(a) npq (b) $np(p + q)$ (c) $np(np + q)$ (d) $np(p + nq)$

17. If $(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n} x^{2n}$, then $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \ldots + a_{2n}^2$ is equal to
(a) 0 (b) 1
(c) a_n (d) None of these

18. If x is very small compared with unity such that $\dfrac{\sqrt{1+x} + \sqrt[3]{(1-x)^2}}{\sqrt{1+x} + (1+x)} = a + bx$, then the values of a and b are
(a) $a = 1, b = \dfrac{5}{6}$ (b) $a = 1, b = -\dfrac{5}{6}$
(c) $a = 1, b = \dfrac{5}{3}$ (d) $a = 1, b = -\dfrac{5}{3}$

19. $7^9 + 9^7$ is divisible by
(a) 128 (b) 24 (c) 64 (d) 72

20. The remainder left out when $8^{2n} - (62)^{2n+1}$ is divided by 9, is
(a) 0 (b) 2 (c) 7 (d) 8

21. For natural numbers m, n, if $(1 - y)^m\ (1 + y)^n = 1 + a_1 y + a_2 y^2 + \ldots$ and $a_1 = a_2 = 10$, then (m, n) is
(a) (20, 45) (b) (35, 20) (c) (45, 35) (d) (35, 45)

22. In the expansion of power of x, if the function $\dfrac{1}{(1 - ax)\ (1 - bx)} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$, then a_n equals
(a) $\dfrac{b^n - a^n}{b - a}$ (b) $\dfrac{a^n - b^n}{b - a}$ (c) $\dfrac{a^{n+1} - b^{n+1}}{b - a}$ (d) $\dfrac{b^{n+1} - a^{n+1}}{b - a}$

23. Coefficient of x^{11} in the expansion of $(1 + x^2)^4\ (1 + x^3)^7\ (1 + x^4)^{12}$ is
(a) 1051 (b) 1106
(c) 1113 (d) 1120

24. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then, n is equal to
(a) 5 (b) 7 (c) 6 (d) 8

25. If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)\ (1 - 2x)^{18}$ in powers of x are both zero, then (a, b) is equal to
(a) $\left(14, \dfrac{272}{3}\right)$ (b) $\left(16, \dfrac{272}{3}\right)$ (c) $\left(16, \dfrac{251}{3}\right)$ (d) $\left(14, \dfrac{251}{3}\right)$

26. For $r = 0, 1, 2, \ldots, 10$, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1 + x)^{10}$, $(1 + x)^{20}$ and $(1 + x)^{30}$. Then, $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to
(a) $B_{10} - C_{10}$ (b) $A_{10}(B_{10}^2 - C_{10}A_{10})$
(c) 0 (d) $C_{10} - B_{10}$

27. The coefficient of x^9 in the expansion of $(1 + x)\ (1 + x^2)\ (1 + x^3)\ \ldots(1 + x^{100})$ is
(a) 2 (b) 6 (c) 9 (d) 8

28. The sum of the last 30 coefficients of powers of x in the binomial expansion of $(1 + x)^{59}$ is
(a) 2^{58} (b) 2^{29} (c) 2^{28} (d) $2^{59} - 2^{29}$

29. The sum of the coefficients of integral powers of x in the binomial expansion of $(1 - 2\sqrt{x})^{50}$ is
(a) $\dfrac{1}{2}(3^{50} - 1)$ (b) $\dfrac{1}{2}(2^{50} + 1)$
(c) $\dfrac{1}{2}(3^{50} + 1)$ (d) $\dfrac{1}{2}(3^{50})$

30. $\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \ldots + \binom{30}{20}\binom{30}{30}$ is equal to
(a) ${}^{30}C_{11}$ (b) ${}^{60}C_{10}$ (c) ${}^{30}C_{10}$ (d) ${}^{65}C_{55}$

31. The larger among $99^{50} + 100^{50}$ and 101^{50}, is
(a) $99^{50} + 100^{50}$ (b) both are equal
(b) $(101)^{50}$ (d) None of these

32. The sum of the series ${}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \ldots + {}^{20}C_{10}$, is
(a) 0 (b) ${}^{20}C_{10}$ (c) ${}^{-20}C_{10}$ (d) $\dfrac{1}{2}\ {}^{20}C_{10}$

33. If $f(n) = \sum_{s=1}^{n} \sum_{r=s}^{n} {}^nC_r\ {}^rC_s$, then $f(3)$ is equal to
(a) 27 (b) 19 (c) 1 (b) 5

Answers

1. (c)	**2.** (a)	**3.** (b)	**4.** (c)	**5.** (a)
6. (b)	**7.** (a)	**8.** (c)	**9.** (a)	**10.** (c)
11. (b)	**12.** (b)	**13.** (a)	**14.** (c)	**15.** (b)
16. (b)	**17.** (c)	**18.** (b)	**19.** (c)	**20.** (b)
21. (d)	**22.** (d)	**23.** (c)	**24.** (c)	**25.** (b)
26. (d)	**27.** (d)	**28.** (a)	**29.** (c)	**30.** (c)
31. (c)	**32.** (d)	**33.** (b)		

STATISTICS

MEASURES OF CENTRAL TENDENCY

An average or a central value of a statistical series is the value of the variable which describes the characteristic of the entire distribution.

Following are commonly used measures of central tendency

(i) Arithmetic Mean (ii) Geometric Mean
(iii) Harmonic Mean (iv) Median
(v) Mode

Arithmetic Mean (AM)

Mean of Ungrouped Data

Let $x_1, x_2, \ldots, x_n$ be n observations, then their arithmetic mean is given by $\bar{x} = \dfrac{x_1 + x_2 + \ldots + x_n}{n} = \dfrac{\sum_{i=1}^{n} x_i}{n}$

Mean of Grouped Data

(i) **Direct method** Let $x_1, x_2, x_3, \ldots, x_n$ be n observations and $f_1, f_2, \ldots, f_n$ be their corresponding frequencies. Then, their arithmetic mean is given by

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots + f_nx_n}{f_1 + f_2 + \ldots + f_n}, \text{ i.e. } \bar{x} = \frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i}$$

(ii) **Assumed mean method** This method helps to reduce the calculation of direct method. In this method, we first find the deviation $d_i = (x_i - A)$ for each observation (term), where A is assumed mean. Then, arithmetic mean is defined as $\bar{x} = A + \dfrac{\Sigma d_i f_i}{\Sigma f_i}$

(iii) **Step deviation method** When the class intervals in a grouped data are equal, then the calculations can be reduce further by taking out the common factor from the deviation.

Here, we first find the deviation $d_i = \dfrac{x_i - A}{h}$ and then arithmetic mean is given by $\bar{x} = A + \dfrac{\Sigma d_i f_i}{\Sigma f_i} \times h$, where A is assumed mean and h is width of class interval.

Weighted Arithmetic Mean

If $w_1, w_2, w_3, \ldots, w_n$ are weights assigned to the values $x_1, x_2, \ldots, x_n$ respectively, then

Weighted arithmetic mean $= \dfrac{w_1x_1 + w_2x_2 + \ldots + w_nx_n}{w_1 + w_2 + \ldots + w_n}$

Combined Mean

If AM of two data sets are $\bar{x}_1$, $\bar{x}_2$ and n_1, n_2 are number of observations in data sets, then

Combined mean $(\bar{x}_{12}) = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$

SPEED BOOSTER

(i) The sum of the deviations of individual values from AM is always zero, i.e. $\sum_{i=1}^{n} (x_i - \bar{x}) = 0$.

(ii) The sum of squares of deviations of individual values from AM is minimum, i.e. $\sum_{i=1}^{n} (x_i - \bar{x})^2 \le \sum_{i=1}^{n} (x_i - A)^2$, where A is any number.

(iii) If each observation is increased (or decreased) by a constant value, then the AM of the observations also increases (or decrease) by the same constant.

(iv) If each observation is multiplied by a number k, then original mean is increased by k times.

Geometric Mean (GM)

GM of Ungrouped Data

If $x_1, x_2, \ldots, x_n$ are n observations, none of them being zero and negative, then their geometric mean is defined as

$$GM = (x_1 \cdot x_2 \ldots x_n)^{1/n}$$

or $GM = \text{antilog}\left(\dfrac{\log x_1 + \log x_2 + \ldots + \log x_n}{n}\right)$

GM of Grouped Data

$$GM = (x_1^{f_1} \cdot x_2^{f_2} \cdot x_3^{f_3} \ldots x_n^{f_n})^{1/N}$$

or

$$GM = \text{antilog}\left(\frac{\sum_{i=1}^{n} f_i \log x_i}{N}\right), \text{ where } N = \sum_{i=1}^{n} f_i$$

SPEED BOOSTER

If $G_1, G_2, \ldots, G_k$ are geometric means of $n_1, n_2, \ldots, n_k$ observations respectively, then the **combined mean** of $n_1 + n_2 + \ldots + n_k = N$ observations can be obtained by the formula

Combined GM $= \text{antilog}\left[\dfrac{1}{N}\sum_{i=1}^{k} (n_i \log G_i)\right]$

Harmonic Mean (HM)

HM of Ungrouped Data

The harmonic mean of n observations $x_1, x_2, x_3, \ldots, x_n$ is defined as

$$\text{HM} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$$

HM of Grouped Data

If $x_1, x_2, x_3, \ldots, x_n$ are n observations which occur with frequencies $f_1, f_2, \ldots, f_n$ respectively, then their HM is given by

$$\text{HM} = \frac{\sum_{i=1}^{n} f_i}{\sum_{i=1}^{n} \left(\frac{f_i}{x_i}\right)}$$

REMEMBER

Relation between AM, GM and HM

Arithmetic mean, Geometric mean and Harmonic mean have following relation :

(i) AM $\geq$ GM $\geq$ HM

(ii) $(\text{GM})^2 = \text{AM} \times \text{HM}$

(iii) AM = GM = HM, when all the observations are equal.

Median

Median is the middle most or the central value of the data, when the observations are arranged either in ascending or in descending order of their magnitudes.

Median of Ungrouped Data

Let n be the number of observations, then following steps are taken to find the median:

***Step* I** Arrange the data in ascending or descending order.

***Step* II** (a) If n is odd, then

$$\text{Median} = \text{Value of } \left(\frac{n+1}{2}\right)\text{th term}$$

(b) If n is even, then

$$\text{Median} = \frac{\left(\frac{n}{2}\right)\text{th term} + \left(\frac{n}{2}+1\right)\text{th term}}{2}$$

Median of Grouped Data

For Discrete Frequency Distribution

Following are the steps to find the median:

***Step* I** Arrange the data in ascending or descending order.

***Step* II** Prepare cumulative frequency table.

***Step* III**

(a) If N is odd, then

$$\text{Median} = \text{Value of } \left(\frac{N+1}{2}\right)\text{th term}$$

(b) If N is even, then

$$\text{Median} = \frac{\left(\frac{N}{2}\right)\text{th term} + \left(\frac{N}{2}+1\right)\text{th term}}{2}$$

where $N = \Sigma f$

For Continuous Frequency Distribution

Following are the steps to find median :

***Step* I** Prepare the cumulative frequency table.

***Step* II** Find the median class i.e. the class in which $\left(\frac{N}{2}\right)$th observation lies.

***Step* III** Median can be obtained by the formula,

$$\text{Median} = l + \left(\frac{\left(\frac{N}{2}\right) - C_f}{f}\right) \times h$$

where, l = lower limit of the median class

N = sum of frequency

f = frequency of the median class

h = length of the median class

C_f = cumulative frequency of the class preceding the median class

REMEMBER

If data is arranged in descending order, then

$$\text{Median} = u - \left(\frac{\frac{N}{2} - C_f}{f}\right) \times h$$

where, u = upper limit of the median class.

Mode

The mode or modal value of data is that observation which occurs with maximum frequency.

Mode of Ungrouped Data

Observation which is repeated maximum number of times is the mode.

Mode of Grouped Data

For Discrete Frequency Distribution

Observation, corresponding to the maximum frequency, is the mode.

For Continuous Frequency Distribution

In this case, mode is calculated by the formula,

$$\text{Mode} = l + \left[\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right] \times h$$

where,

l = lower limit of modal class

f_1 = frequency of the modal class

f_0 = frequency of the class preceding the modal class

f_2 = frequency of the class succeeding the modal class

h = size of the modal class

REMEMBER

Empirical Relationship between Mean, Median and Mode

Mode = 3 Median – 2 Mean

Measures of Dispersion

A measure of dispersion is designed to know the extent to which the individual observation vary from their average. Following are the ways to measure dispersion:

Range, Mean deviation, Standard deviation and Variance

Range

Range is the difference between two extreme observations of distribution. If L and S are largest and smallest observations respectively, then range $= L - S$

and coefficient of range $= \dfrac{L - S}{L + S}$

Note In a grouped frequency distribution, the range is calculated by subtracting the lower limit of lower class interval from the upper limit of the highest class interval.

Mean Deviation

The arithmetic average of the absolute deviation from the mean, median or mode of the terms of the distribution, is called mean deviation.

Mean Deviation of Ungrouped Data

If $x_1, x_2, \ldots, x_n$ are n observations, then mean deviation from average M (mean, mode or median) is

$$\text{MD} = \frac{\sum_{i=1}^{n} |x_i - M|}{n} = \frac{\sum_{i=1}^{n} d_i}{n},$$

where $d_i = |x_i - M|$

Mean Deviation of Grouped Data

If $x_1, x_2, \ldots, x_n$ are n observations and their corresponding frequencies are $f_1, f_2, \ldots, f_n$, then mean deviation from average M is

$$\text{MD} = \frac{\sum_{i=1}^{n} f_i |x_i - M|}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i},$$

where $d_i = |x_i - M|$

REMEMBER

(i) Coefficient of MD $= \dfrac{\text{Mean deviation}}{\text{Corresponding average}}$

(ii) Mean deviation is least, when taken from the median.

Standard Deviation

Standard deviation is the square root of the arithmetic mean of square of deviations of various values from their arithmetic mean. It is generally denoted by σ.

Following are the methods to find standard deviation:

Standard Deviation of Ungrouped Data

(i) **Direct method**

$$\sigma = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n}}$$

where, $\bar{x}$ = arithmetic mean, n = number of observations

(ii) **Shortcut method**

$$\sigma = \sqrt{\left[\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2\right]}$$

where, $d_i = x_i - A$ = deviation from assumed mean A.

Standard Deviation of Grouped Data

(i) **Direct method**

$$\sigma = \sqrt{\frac{\Sigma f_i (x_i - \bar{x})^2}{N}}$$

where, $\bar{x}$ = arithmetic mean

f_i = frequency corresponding to x_i

$N = \Sigma f_i$ = total frequency

(ii) **Shortcut method**

$$\sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$$

where,

$d_i = x_i - A$ = deviation from assumed mean A

f_i = frequency corresponding to x_i

$N = \Sigma f_i$ = total frequency

Coefficient of Standard Deviation

To compare the dispersion of two frequency distributions, the relative measure of standard deviation is computed which is known as coefficient of standard deviation and is given by coefficient of SD = $\sigma/\bar{x}$.

REMEMBER

(i) SD is independent of change of origin but dependent on change of scale, i.e. if $y = ax + b$, then $\sigma_y = |a|\sigma_x$.

(ii) SD is not less than mean deviation from mean.

(iii) Let n_1 and n_2 be the sizes of two distribution. Their means and standard deviations are $\bar{x}_1$, $\bar{x}_2$ and σ_1, σ_2, respectively. If $\bar{x}$ denotes the combined mean of two distribution, then $\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$.

The combined standard deviation of two distribution is given by

$$\sigma = \sqrt{\frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}}$$

or $$\sigma = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1n_2(\bar{x}_1 - \bar{x}_2)^2}{(n_1 + n_2)^2}}$$

where, $d_1 = \bar{x}_1 - \bar{x}$ and $d_2 = \bar{x}_2 - \bar{x}$

Variance

The square of standard deviation is called variance. Following are the methods to find variance:

Variance of Ungrouped Data

$$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}, \text{ where } \bar{x} = \text{arithmetic mean}$$

n = number of observations

Variance of Grouped Data

$$\sigma^2 = \frac{\sum_{i=1}^{n} f_i(x_i - \bar{x})^2}{N} \quad \text{or} \quad \sigma^2 = \frac{\sum_{i=1}^{n} f_id_i^2}{N} - \left(\frac{\sum_{i=1}^{n} f_id_i}{N}\right)^2$$

where, $\bar{x}$ = arithmetic mean

$d_i = x_i - A$ = deviation from assumed mean A

$N = \Sigma f_i$ = total frequency

Combined Variance

If n_1, n_2 are the sizes, $\bar{x}_1, \bar{x}_2$ are the means and σ_1, σ_2 are the standard deviation of two data sets, then combined variance of two data sets is

$$\sigma^2 = \frac{1}{n_1 + n_2}[n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)]$$

where, $d_1 = \bar{x}_1 - \bar{x}, d_2 = \bar{x}_2 - \bar{x}$

and $$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

Coefficient of Variance

Coefficient of variance = Coefficient of SD $\times 100 = \frac{\sigma}{\bar{x}} \times 100$

SPEED BOOSTER

(i) Standard deviation $\leq$ range, also variance $\leq$ (range)^2

(ii) Mean deviation = $\frac{4}{5}$ (standard deviation)

(iii) Standard deviation of first n natural numbers is $\sqrt{\frac{n^2 - 1}{12}}$.

(iv) $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

(v) $2 + 4 + 6 + \ldots + 2n = n(n + 1)$

(vi) $1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}$

(vii) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$

(viii) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n + 1)}{2}\right)^2$

Notice Board

Dates for JEE Main Exam are out, here are the important dates.

- Submission of online application (www.jeemain.in) 01/12/2016 to 02/01/2017.
- Last date of submission of online application is 02/01/2017.
- Test admit card can be downloaded from website and it is available around 2nd week of March 2017 onwards.
- Dates of Examination are as follows :
  - (i) Offline (Pen and paper) mode of exam : 02/04/2017 (Sunday)
    Paper 1 : BE/B.Tech : 9:30 to 12:30
    Paper 2 : B.Arch/B. Pharma : 14:00 to 17:00
  - (ii) Online mode (Computer Based)
    08/04/2017 (Saturday)
    09/04/2017 (Sunday)
- Answer key and OMR sheet are available on website around 18 to 22 April, 2017.
- Declaration of JEE Main paper 1 score/All India rank and top 220000 candidates eligible for JEE advanced : 27/04/2017.
- Declaration of JEE Main paper 2 score : 27/05/2017
- Score/rank cards are available on website.

AADHAAR CARD is compulsory for filling the examination form.

MASTER STROKES

1. The mean of the numbers $\frac{{}^{50}C_0}{1}, \frac{{}^{50}C_2}{3}, \frac{{}^{50}C_4}{5}, \ldots, \frac{{}^{50}C_{50}}{51}$ is equal to

(a) $\frac{2^{50}}{51}$ (b) $\frac{2^{49}}{51}$

(c) $\frac{2^{49}}{39 \times 17}$ (d) None of these

2. If a variate takes the values 0, 1, 4, 9, …, n^2 with frequencies x^n, ${}^nC_1 x^{n-1}y^1$, ${}^nC_2 x^{n-2}y^2$, …, y^n such that $x + y = 1$, then mean is equal to

(a) $nx(nx + y)$ (b) $ny(ny + x)$

(c) $nx(ny + x)$ (d) $ny(nx + y)$

3. The mean (average) of the product of n natural numbers taken two at a time is

(a) $n(n + 1)(3n + 2)$ (b) $\frac{(n + 1)(3n + 1)}{24}$

(c) $\frac{(n + 1)(3n + 2)}{12}$ (d) None of these

4. In a frequency distribution, if the mean and median are 21 and 22 respectively, then its mode is approximately

(a) 24.0 (b) 25.5 (c) 20.5 (d) 22.0

5. If the median of 21 observations is 40 and if the observations greater than the median are increased by 5, then the median of the new data will be

(a) 40 (b) 45

(c) $40 + \frac{50}{21}$ (d) $45 - \frac{50}{21}$

6. The average marks of boys in a class is 52 and that of girls is 42. The average marks of boys and girls combined is 50. The percentage of boys in the class is

(a) 40 (b) 20 (c) 80 (d) 60

7. Median of the odd divisors of 360 is

(a) the mean of 3rd and 4th item

(b) the mean of 4th and 5th item

(c) the mean of 5th and 6th item

(d) None of the above

8. If G is the GM of the product of k, set of observations with GM's $G_1, G_2, G_3, \ldots, G_k$ respectively, then G is equal to

(a) $\log G_1 + \log G_2 + \ldots + \log G_k$

(b) $G_1, G_2, G_3, \ldots, G_k$

(c) $\log G_1, \log G_2, \log G_3, \ldots, \log G_k$

(d) $k \log (G_1, G_2 \ldots G_k)$

9. If the difference between the mode and median is 2, then the difference between the median and mean is (in the given order)

(a) 2 (b) 4 (c) 1 (d) 0

10. In a batch of 13 students, 4 have failed. The marks of the successful candidates were 41, 57, 38, 61, 36, 35, 71, 50 and 40. Then, the median marks are

(a) 40 (b) 50

(c) 41 (d) Cannot be determined

11. The sum of deviation of n observation about 25 is 25 and sum of deviations of the same n observations about 35 is –25. The mean of observation is

(a) 25 (b) 30 (c) 35 (d) 40

12. If the difference between mean and mode is 63, then the difference between mean and median is (in the given order)

(a) 189 (b) 21 (c) 31.5 (d) 48.5

13. Mean of the numbers 1, 2, 3, …, n with respective weights $1^2 + 1, 2^2 + 2, 3^2 + 3, \ldots, n^2 + n$ is

(a) $\frac{3n(n + 1)}{2(2n + 1)}$ (b) $\frac{2n + 1}{3}$

(c) $\frac{3n + 1}{4}$ (d) $\frac{3n + 1}{2}$

14. If the mean of a set of observations $x_1, x_2, x_3, \ldots, x_n$ is $\bar{x}$, then mean of observations $x_i + 3i, \forall\, i = 1, 2, 3, \ldots, n$ equals

(a) $\bar{x} + 3(n + 1)$ (b) $\bar{x} + \frac{3(n + 1)}{2}$

(c) $\bar{x} + \frac{n + 1}{2n}$ (d) None of these

15. The mean of 10 numbers is 12.5. If the mean of the first six numbers is 15 and the last five numbers is 10, then the sixth number is

(a) 15 (b) 12 (c) 18 (d) None of these

16. If G_1, G_2 are the geometric means of two series of observations and G is the GM of the ratios of the corresponding observations, then G is equal to

(a) $\frac{G_1}{G_2}$ (b) $\log G_1 - \log G_2$

(c) $\log G_1 / \log G_2$ (d) $\log(G_1 \cdot G_2)$

17. The mean and median of 100 items are 50 and 52, respectively. The value of largest item is 100. It was later found that it is 110 not 100. The true mean and median are

(a) 50.10, 51.5 (b) 50.10, 52 (c) 50, 52 (d) None of these

18. If a variable takes the discrete values $\alpha + 4, \alpha - \frac{7}{2}, \alpha - \frac{5}{2}, \alpha - 3, \alpha - 2, \alpha + \frac{1}{2}, \alpha - \frac{1}{2}, \alpha + 5 (\alpha > 0)$, then median is

(a) $\alpha - \frac{5}{4}$ (b) $\alpha - \frac{1}{2}$ (c) $\alpha - 2$ (d) $\alpha + \frac{5}{4}$

19. In a factory, workers work in three shifts say shift 1, shift 2 and shift 3 and they get wages in the ratio 4 : 5 : 6 depending on the shifts 1, 2 and 3, respectively. Number of workers in the shifts are in the ratio 3 : 2 : 1. If total number of workers working is 1500 and wages per worker in Ist shift is ₹ 400. Then, mean wage of a worker is

(a) ₹ 467 *(b)* ₹ 500 *(c)* ₹ 600 *(d)* ₹ 400

20. The geometric mean of numbers $7, 7^2, 7^3, \ldots, 7^n$ is

(a) $7^{7/4}$ *(b)* $7^{4/7}$ *(c)* $7^{\frac{n-1}{2}}$ *(d)* $7^{\frac{n+1}{2}}$

21. If the mean deviation about the median of the numbers $a, 2a, \ldots, 50a$ is 50, then $|a|$ is equal to

(a) 5 *(b)* 2

(c) 3 *(d)* 4

22. For two data sets, each of size 5, the variance are given to be 4 and 5 and the corresponding means are given to be 2 and 4, respectively. The variance of combined data set is

(a) 6 *(b)* $\frac{13}{2}$

(c) $\frac{5}{2}$ *(d)* $\frac{11}{2}$

23. The mean of the numbers $a, b, 8, 5, 10$ is 6 and the variance is 6.80. Then, which one of the following gives possible values of a and b?

(a) $a = 0, b = 7$ *(b)* $a = 5, b = 2$

(c) $a = 1, b = 6$ *(d)* $a = 3, b = 4$

24. The mean deviation from mean of the observations $a, a + d, a + 2d, \ldots, a + 2nd$ is

(a) $\frac{n(n+1)a^2}{3}$ *(b)* $\frac{n(n+1)d^2}{2}$

(c) $a + \frac{n(n+1)d^2}{2}$ *(d)* None of these

25. Coefficient of variation of two distribution are 50 and 60 and their arithmetic means are 30 and 25, respectively. The difference of their standard deviation is

(a) 0 *(b)* 1

(c) 1.5 *(d)* 2.5

26. Let a population A has 100 observations 101, 102, ..., 200 and another population B has 100 observations 151, 152, ..., 250. If V_A and V_B represent the variances of the two populations respectively, then $\frac{V_A}{V_B}$ is

(a) $\frac{9}{4}$ *(b)* $\frac{4}{9}$ *(c)* $\frac{2}{3}$ *(d)* 1

27. If the standard deviation of a set of observation is 4 and if each observation is divided by 4, then the standard deviation of the new set of observations will be

(a) 4 *(b)* 3 *(c)* 2 *(d)* 1

28. Consider a set of observations $x_1, x_2, x_3, \ldots, x_{101}$; it being given that $x_1 < x_2 < x_3 < \ldots < x_{100} < x_{101}$; then the mean deviation of this set of observations about a point K is minimum, when K is equal to

(a) x_1 *(b)* x_{51}

(c) $\frac{x_1 + x_2 + \ldots + x_{101}}{101}$ *(d)* x_{50}

29. The variable x takes two values x_1 and x_2 with frequencies f_1 and f_2, respectively. If σ denotes the standard deviation of x, then

(a) $\sigma^2 = \frac{f_1x_1^2 - f_2x_2^2}{f_1 + f_2} + \left(\frac{f_1x_1 - f_2x_2}{f_1 + f_2}\right)^2$

(b) $\sigma^2 = \frac{f_1f_2}{(f_1 + f_2)^2}(x_1 - x_2)^2$

(c) $\sigma^2 = \frac{(x_1 - x_2)^2}{f_1 + f_2}$

(d) None of the above

30. Consider first 10 positive integers having standard deviation 2.87. If we multiply each number by –1 and then add 1 to each number, the standard deviation of the numbers so obtained is

(a) 8.25 *(b)* 2.87 *(c)* –2.87 *(d)* –8.25

31. The sum of square of deviation of 10 observations taken from mean 50 is 250. The coefficients of variation is

(a) 50% *(b)* 10%

(c) 40% *(d)* None of these

32. If the variance of 1, 2, 3, 4, 5, ..., 10 is $\frac{19}{12}$, then the standard deviation of 3, 6, 9, 12, ..., 30 is

(a) $\frac{297}{4}$ *(b)* $\frac{3}{2}\sqrt{33}$

(c) $\frac{3}{2}\sqrt{99}$ *(d)* $\sqrt{\frac{99}{12}}$

33. If $x_1, x_2, \ldots, x_{18}$ are observations such that $\sum_{i=1}^{18}(x_i - 8) = 9$ and $\sum_{i=1}^{18}(x_i - 8)^2 = 45$, then the standard deviation of these observation is

(a) $\sqrt{\frac{51}{34}}$ *(b)* 5 *(c)* $\sqrt{5}$ *(d)* $\frac{3}{2}$

34. The standard deviation of the numbers 31, 32, 33 ,..., 46, 47 is

(a) $\sqrt{\frac{17}{12}}$ *(b)* $\sqrt{\frac{47^2 - 1}{12}}$

(c) $2\sqrt{6}$ *(d)* $4\sqrt{3}$

35. If the standard deviation of the observations $-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$ is $\sqrt{10}$, then standard deviation of the observations 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 will be

(a) $\sqrt{10} + 20$ *(b)* $\sqrt{10} + 10$

(c) $\sqrt{10}$ *(d)* None of these

36. In an experiment with 15 observations of x, the following results were available $\Sigma x^2 = 2830$, $\Sigma x = 170$. An observation that was 20, was found to be wrong and was replaced by the correct value 30. Then, the correct variance is

(a) 78.00 *(b)* 188.66

(c) 177.33 *(d)* 8.33

37. Consider $(2n+1)$ observations $x_1, -x_1, x_2, -x_2, \ldots, x_n, -x_n$ and 0, where x_i's are all distinct. Let SD and MD denote the standard deviation and median, respectively. Then, which of the following is always true?

(a) SD < MD *(b)* SD = MD

(c) SD > MD *(d)* SD = $\frac{1}{2}$ MD

38. The mean and variance of n observations $x_1, x_2, x_3, \ldots, x_n$ are 5 and 0, respectively. If $\sum_{i=1}^{n} x_i^2 = 400$, then the value of n is equal to

(a) 80 *(b)* 25 *(c)* 20 *(d)* 16

39. In a series of $2n$ observations, half of them equal to a and remaining half equal to $-a$. If the standard deviation of the observations is 2, then $|a|$ is equal to

(a) $\frac{\sqrt{2}}{n}$ *(b)* $\sqrt{2}$ *(c)* 2 *(d)* $\frac{1}{n}$

40. Standard deviation of the first $(2n+1)$ natural numbers is equal to

(a) $\sqrt{\frac{n(n+1)}{2}}$ *(b)* $\sqrt{\frac{n(n+1)(2n+1)}{3}}$

(c) $\sqrt{\frac{n(n+1)}{3}}$ *(d)* $\sqrt{\frac{n(n-1)}{2}}$

41. If the variable takes the values 0, 1, 2, …, n with frequencies proportional to the binomial coefficients $C(n, 0), C(n, 1), C(n, 2), \ldots, C(n, n)$ respectively, then the variance of the distribution is

(a) n *(b)* $\frac{\sqrt{n}}{2}$ *(c)* $\frac{n}{2}$ *(d)* $\frac{n}{4}$

42. Let r be the range and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ be the SD of a set of observations $x_1, x_2, \ldots, x_n$, then

(a) $S \le r\sqrt{\frac{n}{n-1}}$ *(b)* $S = r\sqrt{\frac{n}{n-1}}$

(c) $S \ge r\sqrt{\frac{n}{n-1}}$ *(d)* None of these

43. If the mean deviation of the numbers $1, 1+d, 1+2d, \ldots, 1+100d$ from their mean is 255, then d is equal to

(a) 10.0 *(b)* 20.0

(c) 10.1 *(d)* 20.2

44. A sample of 35 observations has the mean 80 and SD as 4. A second sample of 65 observations from the same population has mean 70 and SD is 3. Then, the SD of the combined sample is

(a) 5.85 *(b)* 5.58

(c) 34.2 *(d)* None of these

45. One set containing five numbers has mean 8 and variance 24 and the second set containing three numbers has mean 8 and variance 24. Then, the variance of the combined set is

(a) 42 *(b)* 24 *(c)* 20 *(d)* 25

46. The standard deviation of 25 numbers is 40. If each of the numbers which is greater than the standard deviation is increased by 5, then the new standard deviation will be

(a) 40 *(b)* 45

(c) $40 + \frac{21}{25}$ *(d)* None of these

47. If the standard deviation of a variable x is σ, then the standard deviation of variable $\frac{ax+b}{c}$ is

(a) $a\sigma$ *(b)* $\frac{a}{c}\sigma$

(c) $\left|\frac{a}{c}\right|\sigma$ *(d)* $\frac{a\sigma + b}{a}$

48. The standard deviation of some temperature data in °C is 5. If the data were converted into °F, then the variance would be

(a) 81 *(b)* 57 *(c)* 36 *(d)* 25

49. The mean of 100 observations is 50 and their standard deviation is 5. The sum of the squares of observations is

(a) 50000 *(b)* 250000

(c) 252500 *(d)* 255000

50. A scientist is weighing each of 30 fishes. Their mean weight worked out is 30 g and standard deviation is 2 g. Later, it was found that the measuring scale was misaligned and always under reported every fish weight by 2 g. The correct mean and standard deviation (in g) of weight of fishes are respectively

(a) 32, 4 *(b)* 28, 2 *(c)* 28, 4 *(d)* 32, 2

Answers

1. (c)	**2.** (b)	**3.** (c)	**4.** (a)	**5.** (a)
6. (c)	**7.** (a)	**8.** (b)	**9.** (c)	**10.** (c)
11. (b)	**12.** (b)	**13.** (c)	**14.** (b)	**15.** (a)
16. (a)	**17.** (b)	**18.** (a)	**19.** (a)	**20.** (d)
21. (d)	**22.** (d)	**23.** (c)	**24.** (d)	**25.** (a)
26. (d)	**27.** (d)	**28.** (b)	**29.** (b)	**30.** (b)
31. (b)	**32.** (b)	**33.** (d)	**34.** (c)	**35.** (c)
36. (a)	**37.** (c)	**38.** (d)	**39.** (c)	**40.** (c)
41. (d)	**42.** (a)	**43.** (c)	**44.** (a)	**45.** (b)
46. (a)	**47.** (c)	**48.** (a)	**49.** (c)	**50.** (d)

TARGET JEE 2017

TEST RIDER

Comprehensive Simulator Test Series for JEE Main & Advanced

JEE MAIN

(PAPER ONE PREP UP)

FULL TEST 1

(COMPLETE SYLLABUS)
FULLY SOLVED

Mock Questions from Complete Syllabus with Complete Solutions

INSTRUCTIONS

- This test consists of 30 questions and each question is allotted 4 marks for correct response.
- Candidates will be awarded marks as stated above for correct response of each question. 1/4 mark will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
- There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

1. If $\begin{vmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 6 & 1 & c \end{vmatrix} = 8\cos^3\theta - 4\cos\theta + 6$, then the value of θ is

(a) $\cos^{-1}\frac{c}{2}$ (b) $\cos^{-1} c$ (c) $\cos\frac{c}{2}$ (d) $\frac{c}{2}$

2. Die A has 4 red and 2 white faces whereas die B has 2 red and 4 white faces. A coin is flipped once. If it shows a head, the game continues by throwing die A. If it shows tail, then die B is to be used. If the probability that die A is used is 32/33 when it is given that red turns up every time in first n throws, then the value of n is

(a) 6 (b) 4 (c) 3 (d) 5

3. In an examination, the maximum marks for each of the three papers is 50 and the maximum marks for the fourth paper is 100. Then, the number of ways in which the candidate can score 605 marks in aggregate is equal to

(a) 110551 (b) 112024
(c) 103231 (d) 405321

4. Consider the system of equations in x, y and z as

$x\sin 3\theta - y + z = 0$, $x\cos 2\theta + 4y + 3z = 0$

$2x + 7y + 7z = 0$

If this system has a non-trivial solution and also $\theta = \left(2 + \frac{(-1)^n}{6}\right)$, then the value of n is

(a) 3 (b) 1/2 (c) 2 (d) 5

5. The sum to n terms of the series $\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \ldots$ is $\frac{n^2+n}{k(n^2+n+1)}$. Then, the value of k is

(a) 1/2 (b) 3 (c) 2/3 (d) 2

6. If f and g are continuous functions on $[0, a]$ satisfying $f(x) = f(a-x)$ and $g(x) + g(a-x) = 2$, then $\int_0^a f(x)\, g(x)\, dx$ is equal to

(a) $\int_a^0 f(x)\, dx$ (b) $\int_0^a 2f(x)\, dx$
(c) $\int_0^{2a} f(x)\, dx$ (d) $\int_0^a f(x)\, dx$

7. The value of $\int \frac{\cos^4 x}{\sin^3 x\,(\sin^5 x + \cos^5 x)^{3/5}}\, dx$ is

(a) $-\frac{1}{2}(1+\cot^5 x)^{2/5} + C$ (b) $\frac{1}{2}(1+\cot^4 x)^{3/5} + C$
(c) $\frac{1}{3}(2+\tan^5 x)^{2/5} + C$ (d) $\frac{1}{3}(1+\tan^5 x)^{3/5} + C$

8. If the area bounded by $y = x^2 + 2x - 3$ and the line $y = kx + 1$ is the least, then the least area is

(a) $\frac{32}{3}$ sq units (b) $\frac{34}{3}$ sq units
(c) $\frac{31}{2}$ sq units (d) $\frac{29}{3}$ sq units

9. If $xdy = y(dx + ydy)$, $y(1) = 1$ and $y(x) > 0$. Then, $y(-3)$ is equal to

(a) 3 (b) 2 (c) 4 (d) 6

10. If pth, qth and rth terms of AP $a_1, a_2, a_3, \ldots$ are in GP, then $a_p : a_q$ is equal to
(a) $\frac{q-p}{r-p}$ *(b)* $\frac{r-q}{q-p}$ *(c)* $\frac{q-p}{r-q}$ *(d)* None of these

11. The value of
$$\lim_{x\to\infty} 2\left\{\frac{1}{2x+1}+\frac{1}{3(2x+1)^3}+\frac{1}{5(2x+1)^5}+\ldots\right\}$$
is equal to
(a) 0 *(b)* 1 *(c)* 1/2 *(d)* 2

12. Five balls of different colours are to be placed in three boxes of different sizes. Each box can hold all five. In how many different ways can we place the balls so that no box remain empty?
(a) 150 *(b)* 130 *(c)* 243 *(d)* 246

13. Let z_1, z_2 and z_3 represent the vertices A, B and C of the ΔABC in the argand plane, such that $|z_1| = |z_2| = |z_3| = 5$. Then,
$z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C$ equals
(a) 0 *(b)* $\frac{z_1}{z_2}$ *(c)* 1 *(d)* $\frac{1}{2}$

14. Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $S = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then,
(a) $P \subset S$ and $S - P = \phi$ *(b)* $S \not\subset P$
(c) $P \not\subset S$ *(d)* $P = S$

15. If $x^2 - 10ax - 11b = 0$ has roots c and d and $x^2 - 10cx - 11d = 0$ has roots a and b, then $2(b+d) - (c+a)$ is equal to
(a) 2156 *(b)* 1989 *(c)* 2057 *(d)* 2346

16. If $f(x) = \begin{cases} xe^{-\left\{\frac{1}{|x|}+\frac{1}{x}\right\}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f(x)$ is
(a) discontinuous at $x = 0$
(b) not differentiable at $x = 0$
(c) differentiable at $x = 0$
(d) None of the above

17. If $A(4, -3)$ and $B(-4, 2)$ are vertices of a ΔABC. Also, the centroid of this triangle moves on the line $2x + 3y = 1$, then the locus of vertex C is the line
(a) $2h + 3k = 6$ *(b)* $2h + 2k = 3$
(c) $h + 3k = 1$ *(d)* $h + 2k = 1$

18. If the tangent at the point P on the circle $x^2 + y^2 + 8x + 8y = 8$ meets the straight line $3x - 2y + 6 = 0$ at a point Q on the Y-axis, then the length of PQ is
(a) 3 *(b)* $3\sqrt{5}$ *(c)* 6 *(d)* 5

19. Equation of the ellipse whose axes are the axes of coordinates and which passes through the point $(-3, 1)$ and has eccentricity $\sqrt{\frac{2}{5}}$, is
(a) $5x^2 + 3y^2 = 32$ *(b)* $6x^2 + 5y^2 = 33$
(c) $3x^2 + 5y^2 - 15 = 0$ *(d)* $5x^2 + 3y^2 - 32 = 0$

20. The tangents at the end points of any chord through $(1,0)$ to the parabola $y^2 + 4x = 8$ intersect at
(a) 45° on $x - 3 = 0$ *(b)* 45° on $x + 3 = 0$
(c) 90° on $x + 3 = 0$ *(d)* 90° on $x - 3 = 0$

21. If the ratio of the length of the two unequal sides and the base of a triangle are given, then the vertex lies on a/an
(a) ellipse *(b)* circle *(c)* straight line *(d)* parabola

22. Negation of the statement $p \to (q \wedge r)$ is
(a) $\sim p \to \sim(q \vee r)$ *(b)* $\sim p \to \sim(q \wedge r)$
(c) $(q \wedge r) \to p$ *(d)* $p \wedge (\sim q \vee \sim r)$

23. If 162, 163, 160, 164, 160, 170, 161, 164 are the heights (in centimetres) of 8 students, then their standard deviation is
(a) 3 cm *(b)* 3.02 cm *(c)* 3.04 cm *(d)* 3.06 cm

24. If the straight lines $\frac{x-1}{k} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x-2}{3} = \frac{y-3}{k} = \frac{z-1}{2}$ intersect at a point, then the integer k is equal to
(a) -2 *(b)* -5 *(c)* 5 *(d)* 2

25. The equation of the plane through the intersection of the planes $x + y + z = 1$ and $2x + 3y - z + 4 = 0$ and parallel to X-axis, is
(a) $y - 3z + 6 = 0$ *(b)* $y + 3z + 6 = 0$
(c) $3y - z + 6 = 0$ *(d)* $3y - 2z + 6 = 0$

26. A vector of magnitude 12 units perpendicular to the plane containing the vectors $4\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} + 8\hat{\mathbf{j}} + \hat{\mathbf{k}}$, is
(a) $-5\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 8\hat{\mathbf{k}}$ *(b)* $8\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 8\hat{\mathbf{k}}$
(c) $5\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 8\hat{\mathbf{k}}$ *(d)* $8\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 8\hat{\mathbf{k}}$

27. A spherical iron ball of radius 10 cm, coated with a layer of ice of uniform thickness melts at a rate of 400π cm^3/min. The rate at which the thickness of decrease when the thickness of ice is 5 cm, is
(a) 4 cm/min *(b)* 2 cm/min
(c) $\frac{1}{376}$ cm/min *(d)* 5 cm/min

28. The expression $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$ is equal to
(a) 7 *(b)* 0 *(c)* 1 *(d)* None of these

29. The value of
$$\sin\left(\sin^{-1}\frac{1}{3} + \sec^{-1}3\right) + \cos\left(\tan^{-1}\frac{1}{2} + \tan^{-1}2\right) \text{ is}$$
(a) 1 *(b)* 2 *(c)* 3 *(d)* 4

30. If $f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2 - x, & \text{for } 1 \le x < 2. \\ x - (1/2)x^2, & \text{for } x = 2 \end{cases}$
Then, $f'(1)$ is equal to
(a) -1 *(b)* 1
(c) 0 *(d)* None of these

Answers with Explanation

1. *(a)* Given, $\begin{vmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 6 & 1 & c \end{vmatrix} = 8\cos^3\theta - 4\cos\theta + 6$

Taking LHS $= \begin{vmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 6 & 1 & c \end{vmatrix}$

Applying $C_1 \to C_1 - cC_2$

$$\text{LHS} = \begin{vmatrix} 0 & 1 & 0 \\ 1-c^2 & c & 1 \\ 6-c & 1 & c \end{vmatrix}$$

Expanding along R_1

$$\text{LHS} = -\begin{vmatrix} 1-c^2 & 1 \\ 6-c & c \end{vmatrix}$$

$$= -(c - c^3 - 6 + c)$$

$$= c^3 - 2c + 6$$

$\therefore \quad c^3 - 2c + 6 = 8\cos^3\theta - 4\cos\theta + 6$

$\Rightarrow \quad c^3 - 2c = 8\cos^3\theta - 4\cos\theta$

On comparing, we get

$$c^3 = (2\cos\theta)^3 \Rightarrow c = 2\cos\theta$$

$$\therefore \quad \theta = \cos^{-1}\frac{c}{2}$$

2. *(d)* Let R be the event that a red face appears in each of the first n throws.

E_1 : Die A is used when head has already fallen

and E_2 : Die B is used when tail has already fallen

$$\therefore \quad P(R/E_1) = \left(\frac{2}{3}\right)^n \text{ and } P\left(\frac{R}{E_2}\right) = \left(\frac{1}{3}\right)^n$$

As per the given condition,

$$\frac{P(E_1)P(R/E_1)}{P(E_1)\,P(R/E_1) + P(E_2)\,P(R/E_2)} = \frac{32}{33}$$

$$\Rightarrow \quad \frac{\frac{1}{2}\left(\frac{2}{3}\right)^n}{\frac{1}{2}\left(\frac{2}{3}\right)^n + \frac{1}{2}\left(\frac{1}{3}\right)^n} = \frac{32}{33}$$

$$\Rightarrow \quad \frac{2^n}{2^n + 1} = \frac{32}{33} \Rightarrow n = 5$$

3. *(a)* Let the marks scored by the candidate in four papers be x_1, x_2, x_3 and x_4. Then, $x_1 + x_2 + x_3 + x_4 = 150$ (i.e. 60% of 250 is 150), where $0 \le x_1, x_2, x_3 \le 50$, $0 \le x_4 \le 100$

The number of solutions of the equation is same as the coefficient of x^{150} in

$$(1 + x + x^2 + \ldots + x^{50})^3 (1 + x + x^2 + \ldots + x^{100})$$

which is given by

$$\text{Coefficient of } x^{150} \text{ in } \left(\frac{1-x^{51}}{1-x}\right)\left(\frac{1-x^{101}}{1-x}\right)$$

$$= \text{Coefficient of } x^{150} \text{ in } (1-x^{51})^3 (1-x^{101})(1-x)^{-4}$$

$$= \text{Coefficient of } x^{150} \text{ in } (1 - 3x^{51} + 3x^{102} - x^{101})(1-x)^{-4}$$

$$= {}^{153}C_3 - 3 \times {}^{102}C_3 + 3 \times {}^{51}C_3 - {}^{52}C_3 = 110551$$

4. *(c)* Since, given system of equations has a non-trivial solution.

$$\therefore \quad \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

$$\Rightarrow \quad 7\sin 3\theta + 14\cos 2\theta - 14 = 0$$

$$\Rightarrow \quad 3\sin\theta - 4\sin^3\theta + 2 - 4\sin^2\theta - 2 = 0$$

$$\Rightarrow \quad \sin\theta(4\sin^2\theta + 4\sin\theta - 3) = 0$$

$$\Rightarrow \quad \sin\theta\,(2\sin\theta + 3)\,(2\sin\theta - 1) = 0$$

$$\Rightarrow \quad \sin\theta = 0 \text{ or } \sin\theta = \frac{1}{2} \Rightarrow \theta = n\pi$$

$$\text{or} \quad \theta = n\pi + (-1)^n \frac{\pi}{6} \Rightarrow \theta = \left(n + \frac{(-1)^n}{6}\right)\pi$$

$$\because \quad \theta = \left(2 + \frac{(-1)^2}{6}\right)\pi$$

On comparing, we get $n = 2$

5. *(d)* Let T be the rth term of the given series.

Then,

$$T_r = \frac{r}{1 + r^2 + r^4};\ r = 1, 2, 3, \ldots, n$$

$$= \frac{r}{(r^2 + r + 1)(r^2 - r + 1)}$$

$$= \frac{1}{2}\left[\frac{1}{r^2 - r + 1} - \frac{1}{r^2 + r + 1}\right]$$

Therefore, sum of the series is

$$\sum_{r=1}^{n} T_r = \frac{1}{2}\left[\sum_{r=1}^{n}\left(\frac{1}{r^2 - r + 1} - \frac{1}{r^2 + r + 1}\right)\right]$$

$$= \frac{1}{2}\left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{13}\right) + \ldots + \left(\frac{1}{n^2 - n + 1} - \frac{1}{n^2 + n + 1}\right)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{n^2 + n + 1}\right] = \frac{n^2 + n}{2(n^2 + n + 1)}$$

$\therefore \quad k = 2$

6. *(d)* $\int_0^a f(x)\,g(x)\,dx$

$$= \int_0^a f(a - x)\,g(a - x)\,dx$$

$$= \int_0^a f(x)\cdot\{2 - g(x)\}\,dx$$

$$= 2\int_0^a f(x)\,dx - \int_0^a f(x)\,g(x)\,dx$$

$$\Rightarrow \quad 2\int_0^a f(x)\,g(x)\,dx = 2\int_0^a f(x)\,dx$$

$$\Rightarrow \quad \int_0^a f(x)\,g(x)\,dx = \int_0^a f(x)\,dx$$

7. *(a)* Let $I = \int \frac{\cos^4 x}{\sin^3 x(\sin^5 x + \cos^5 x)^{3/5}}\,dx$

$$= \int \frac{\cos^4 x}{\sin^6 x(1 + \cot^5 x)^{3/5}}\,dx$$

$$= \int \frac{\sec^2 x}{\tan^6 x\left(1 + \frac{1}{\tan^5 x}\right)^{3/5}}\,dx$$

Let $\tan x = p$, then $\sec^2 x\, dx = dp$

$$\Rightarrow \quad I = \int \frac{dp}{p^6\left(1+\frac{1}{p^5}\right)^{3/5}}$$

Let $\left(1+\frac{1}{p^5}\right) = k \Rightarrow -5\frac{1}{p^6}\,dp = dk$

$$\therefore \quad I = -\frac{1}{5}\int (k)^{-3/5}\,dk = -\frac{1}{5}(k^{2/5})\,(5/2) + C$$

$$\Rightarrow \quad I = -\frac{1}{2}\left[\frac{p^5+1}{p^5}\right]^{2/5} = -\frac{1}{2}\left[\frac{\tan^5 x + 1}{\tan^5 x}\right]^{2/5}$$

$$= -\frac{1}{2}(1+\cot^5 x)^{2/5} + C$$

8. ***(a)*** Since, x_1 and x_2 are the roots of the equation.

$$\therefore \quad x^2 + 2x - 3 = kx + 1$$

$$\Rightarrow \quad x^2 + (2-k)\,x - 4 = 0$$

$$\Rightarrow \quad \left.\begin{aligned} x_1 + x_2 &= k-2 \\ x_1\,x_2 &= -4 \end{aligned}\right\}$$

$$\sigma = \int_{x_1}^{x_2}[(kx+1) - (x^2+2x-3)]\,dx$$

$$= \left[(k-2)\frac{x^2}{2} - \frac{x^3}{3} + 4x\right]_{x_1}^{x_2}$$

$$= \left[(k-2)\,\frac{x_2^2 - x_1^2}{2} - \frac{1}{3}(x_2^3 - x_1^3) + 4\,(x_2 - x_1)\right]$$

$$= (x_2 - x_1)\left[\frac{(k-2)^2}{2} - \frac{1}{3}\{(x_2+x_1)^2 - x_1x_2\} + 4\right]$$

$$= \sqrt{(x_2+x_1)^2 - 4x_1x_2}\left[\frac{(k-2)^2}{2} - \frac{1}{3}\{(k-2)^2+4\} + 4\right]$$

$$= \frac{\sqrt{(k-2)^2+16}}{6}\left[\frac{1}{6}(k-2)^2 + \frac{8}{3}\right]$$

$$= \frac{[(k-2)^2+16]^{3/2}}{6}$$

which is least when $k = 2$

$$\therefore \text{ Area } (k)_{\text{least}} = \frac{[(2-2)^2+16]^{3/2}}{6} = \frac{32}{3} \text{ sq units}$$

9. ***(a)*** We have,

$$xdy = y(dx + ydy)$$

$$\Rightarrow \quad xdy - ydx = y^2dy$$

$$\Rightarrow \quad -\left(\frac{ydx - xdy}{y^2}\right) = dy \Rightarrow -d\left(\frac{x}{y}\right) = dy$$

On integrating, we get

$$\frac{-x}{y} = y + C \qquad \ldots(i)$$

It is given that $y(1) = 1$, i.e. $y = 1$ when $x = 1$

$$\therefore \quad -1 = 1 + C \quad \Rightarrow \quad C = -2$$

On putting $C = -2$ in Eq. (i), we get

$$\frac{-x}{y} = y - 2$$

On putting $C = -2$ in Eq. (i), we get

$$\frac{-x}{y} = y - 2$$

On putting $x = -3$, we get

$$\frac{3}{y} = y - 2$$

$$\Rightarrow \quad y^2 - 2y - 3 = 0 \Rightarrow (y-3)\,(y+1) = 0$$

$$\Rightarrow \quad y = 3 \qquad [\because y > 0]$$

10. ***(c)*** It is given that a_p, a_q, a_r are in GP.

$$\Rightarrow \quad \frac{a_q}{a_p} = \frac{a_r}{a_q}$$

$$\Rightarrow \quad \frac{a_1 + (q-1)d}{a_1 + (p-1)d} = \frac{a_1 + (r-1)d}{a_1 + (q-1)d}$$

$$\Rightarrow \quad \frac{(q-p)\,d}{a_1 + (p-1)\,d} = \frac{(r-q)\,d}{a_1 + (q-1)\,d}$$

[subtracting 1 from both sides]

$$\Rightarrow \quad \frac{q-p}{a_p} = \frac{r-q}{a_q} \Rightarrow \frac{a_p}{a_q} = \frac{q-p}{r-q}$$

11. ***(a)*** We have,

$$\lim_{x\to\infty} 2\left\{\frac{1}{2x+1} + \frac{1}{3(2x+1)^3} + \frac{1}{5(2x+1)^5} + \ldots\right\}$$

$$= \lim_{x\to\infty} \log\left(\frac{1+\frac{1}{2x+1}}{1-\frac{1}{2x+1}}\right) = \log\left(\frac{2x+2}{2x}\right)$$

$$= \lim_{x\to\infty} \log\left(\frac{x+1}{x}\right) = \lim_{x\to\infty} \log \frac{x\left(1+\frac{1}{x}\right)}{x}$$

$$= \log 1 = 0$$

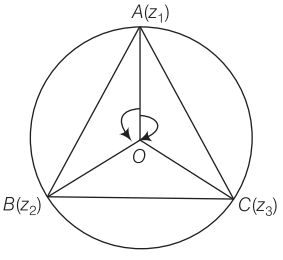
12. ***(a)*** We have balls of different colours and three boxes of different sizes sufficient to hold all five balls.

$\therefore$ Number of ways in which balls could be distributed so that none is empty are (2, 2, 1) or (1, 1, 3).

$$= ({}^5C_2 \times {}^3C_2 \times {}^1C_1 + {}^5C_3 \times {}^2C_1 \times {}^1C_1) \times \frac{3!}{2!}$$

$$= (10 \times 3 \times 1 + 10 \times 2 \times 1) \times 3 = 150$$

13. ***(a)***



$$|z_1| = |z_2| = |z_3| = 5$$

$$\Rightarrow \quad |z| = 5 \text{ is the circumcircle of } \Delta ABC.$$

$$\Rightarrow \angle AOB = 2C, \angle BOC = 2A \text{ and } \angle COA = 2B$$

We have, $\quad \frac{z_2}{z_1} = \frac{OB}{OA}\,e^{i2C} = e^{i2C}$

Similarly, $\quad \frac{z_3}{z_1} = \frac{OC}{OA}\,e^{-i2B} = e^{-i2B}$

Now, $z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C$

$$= z_1\left(\sin 2A + \frac{z_2}{z_1}\sin 2B + \frac{z_3}{z_1}\sin 2C\right)$$

$$= z_1\left(\sin 2A + \frac{z_2}{z_1}\sin 2B + \frac{z_3}{z_1}\sin 2C\right)$$

$$= z_1(\sin 2A + \sin 2B\, e^{i2C} + \sin 2C\, e^{-2iB})$$

$= z_1(\sin 2A + \sin 2B\cos 2C + i\sin 2B\sin 2C$
$+ \sin 2C\cos 2B - i\sin 2C\sin 2B)$
$= z_1\{\sin 2A + \sin(2B + 2C)\}$
$= z_1\{\sin 2A + \sin(2\pi - 2A)\} = 0$

14. ***(d)*** $\because \theta \in P$

$\Rightarrow \sin\theta - \cos\theta = \sqrt{2}\cos\theta$
$\Rightarrow (\sin\theta - \cos\theta)^2 = 2\cos^2\theta$
$\Rightarrow \sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta = 2\cos^2\theta$
$\Rightarrow \cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta = 2\sin^2\theta$
$\Rightarrow (\cos\theta + \sin\theta)^2 = 2\sin^2\theta$
$\Rightarrow \cos\theta + \sin\theta = \sqrt{2}\sin\theta$
$\Rightarrow \theta \in S$
Hence, $P = S$

15. ***(c)*** It is given that c and d are roots of $x^2 - 10ax - 11b = 0$ and a and b are roots of $x^2 - 10cx - 11d = 0$

$\therefore c + d = 10a$ and $a + b = 10c$
$\Rightarrow c + d + a + d = 10(a + c)$ and $c + d - a - b = 10a - 10c$
$\Rightarrow b + d = 9(a + c)$ and $b - d = 11(c - a)$ ...(i)
Again, c is a root of $x^2 - 10ax - 11b = 0$ and a is a root of $x^2 - 10cx - 11d = 0$
$\therefore c^2 - 10ac - 11b = 0$ and $a^2 - 10ac - 11d = 0$
$\Rightarrow (c^2 - a^2) = 11(b - d)$
$\Rightarrow (c - a)(c + a) = 11(b - d)$
$\Rightarrow (c + a) = 14$
$[\because b - d = 11(c - a)$, from Eq. (i)$]$
$\Rightarrow (b + d) = 1089$
$[\because b + d = 9(a + c)$, from Eq. (i)$]$
$\therefore 2(b + d) = 2 \times 1089 = 2178$
Now, $2(b + d) - (c + a) = 2178 - 121 = 2057$

16. ***(b)*** $f'(0^+) = \lim\limits_{x\to 0}\dfrac{f(x) - f(0)}{x}$

$= \lim\limits_{x\to 0}\dfrac{xe^{-2/x}}{x} = \lim\limits_{x\to 0}\dfrac{1}{e^{2/x}} = 0$

$\Rightarrow f'(0^-) = \lim\limits_{x\to 0}\dfrac{f(x) - f(0)}{x} = \lim\limits_{x\to 0}\dfrac{x - 0}{x} = 1$

Here, two values are different.
$\therefore f(x)$ is not differentiable at $x = 0$.

17. ***(a)*** Let (h, k) be the coordinates of the third vertex C.
Then, the coordinates of the centroid G are

$$G\left(\frac{h + 4 - 4}{3}, \frac{k - 3 + 2}{3}\right) = G\left(\frac{h}{3}, \frac{k - 1}{3}\right)$$

It lies on $2x + 3y = 1$

$\therefore 2 \times \dfrac{h}{3} + 3 \times \left(\dfrac{k - 1}{3}\right) = 1$

$\Rightarrow \dfrac{2h}{3} + k - 1 = 1$

$\Rightarrow 2h + 3k - 3 = 3$

$\therefore 2h + 3k = 6$

18. ***(d)*** The line $3x - 2y + 6 = 0$ cuts Y-axis at $Q(0, 3)$. Clearly, PQ is the length of the tangent drawn from Q on the circle

$$x^2 + y^2 + 8x + 8y = 8$$

$\therefore PQ = \sqrt{0 + 9 + 8 \times 0 + 8 \times 3 - 8} = 5$

19. ***(a)*** Let the equation of the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and e be its eccentricity.

It passes through $(-3, 1)$ and has eccentricity, $e = \sqrt{\dfrac{2}{5}}$

$\therefore \dfrac{4}{a^2} + \dfrac{9}{b^2} = 1$ and $e^2 = \dfrac{2}{5}$

$\dfrac{4}{a^2} + \dfrac{9}{a^2(1 - e^2)} = 1$ and $e^2 = \dfrac{2}{5}$

$\dfrac{9}{a^2} + \dfrac{5}{3a^2} = 1 \Rightarrow \dfrac{32}{3} = a^2$

Now, $b^2 = a^2(1 - e^2)$

$\Rightarrow b^2 = \dfrac{32}{3}\left(1 - \dfrac{2}{5}\right) = \dfrac{32}{5}$

Hence, the equation of the ellipse is $5x^2 + 3y^2 = 32$.

20. ***(d)*** We have,

$y^2 + 4x = 8 \Rightarrow y^2 = -4(x - 2)$
$\Rightarrow (y - 0)^2 = -4(x - 2)$
The coordinates of the focus of this parabola are
$(x - 2 = -1, y - 0 = 0)$, i.e. $(x = 1, y = 0)$
The equation of the directrix is
$x - 2 = 1$, i.e. $x - 3 = 0$
We know that the tangents at the end points of a focal chord intersect at 90° on the directrix.
Hence, the tangents intersect at 90° on $x - 3 = 0$.

21. ***(b)*** Let the base of the triangle be the line segment joining $B(0, 0)$ and $C(a, 0)$ and let the vertex be $A(h, k)$, where a is fixed. Also, let

$\dfrac{AB}{AC} = \lambda, \lambda \neq 1$

$\Rightarrow AB^2 = \lambda^2 AC^2$
$\Rightarrow h^2 + k^2 = \lambda^2\{(h - a)^2 + k^2\}$
$\Rightarrow h^2(1 - \lambda^2) + k^2(1 - \lambda^2) + 2a\lambda^2 h - a^2\lambda^2 = 0$
So, $A(h, k)$ lies on

$$x^2 + y^2 + 2a\frac{\lambda^2}{1 - \lambda^2}x - \frac{a^2\lambda^2}{1 - \lambda^2} = 0,$$ which is a circle.

22. ***(d)*** We know that $\sim(p \to q) \equiv p \wedge \sim q$

$\therefore \sim(p \to (q \wedge r)) \equiv p \wedge (\sim(q \wedge r)) \equiv p \wedge (\sim q \vee \sim r)$
[by De-Morgan's law]

23. ***(c)*** We have,

$$\text{Mean}(\bar{x}) = \frac{160 + 160 + 161 + 162 + 163 + 164 + 164 + 170}{8}$$
$= 163$

Height (in cm) x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
160	− 3	9
160	− 3	9
161	− 2	4
162	− 1	1
163	0	0
164	1	1
164	1	1
170	7	49
Total		74

To be Continued at Page 43

TARGET JEE 2017

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JEE MAIN

(PAPER TWO SCALE UP)

FULL TEST 2

(COMPLETE SYLLABUS) WITH ANSWERS

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INSTRUCTIONS

- This test consists of 30 questions and each question is allotted 4 marks for correct response.
- Candidates will be awarded marks as stated above for correct response of each question. 1/4 mark will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
- There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

1. The eccentricity of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose latusrectum is $\frac{1}{4}$ th of its minor axis, is

(a) $\frac{\sqrt{15}}{2}$ (b) $\frac{\sqrt{15}}{4}$

(c) $\frac{\sqrt{25}}{4}$ (d) None of these

2. If $y = \text{cosec}\,(\cot^{-1} x)$, then $\frac{dy}{dx}$ at $x = 3$ is equal to

(a) $\frac{1}{\sqrt{10}}$ (b) $\frac{3}{\sqrt{10}}$ (c) $\frac{1}{\sqrt{5}}$ (d) $\frac{2}{\sqrt{5}}$

3. The equation of a circle with centre at origin and passing through the vertices of an equilateral triangle whose median is of length $3k$, is

(a) $x^2 + y^2 = k^2$ (b) $x^2 + y^2 = 2k^2$

(c) $x^2 + y^2 = 4k^2$ (d) $x^2 + y^2 = \frac{k^2}{2}$

4. $\int_0^{\infty} e^{-3x}(\sin 3x + \cos 3x)\,dx$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) 3 (d) $\frac{1}{3}$

5. The value of $\cos\left[\tan^{-1}\left(\frac{1-x^2}{2x}\right)+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$ is

(a) 0 (b) 1 (c) 2 (d) $\frac{1}{2}$

6. If a, b, c are in AP, then $\frac{1}{\sqrt{a}+\sqrt{b}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{b}+\sqrt{c}}$ are in

(a) AGP (b) GP

(c) HP (d) AP

7. If $\tan\alpha = t\tan\beta$, then $\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}$ is equal to

(a) $\frac{1-t}{1+t}$ (b) $\frac{t-1}{t+1}$

(c) $\frac{1+t}{1-t}$ (d) $\frac{1+t}{t-1}$

8. If tangents at P and R on the parabola $y^2 = 4ax$ intersect at point Q, then ordinates of P, Q and R are

(a) always GP (b) always HP

(c) always AP (d) None of these

9. A line $4x + y = 1$ through the point $P(2, -7)$ meets the line QR whose equation is $3x - 4y + 1 = 0$ at the point Q. If $PQ = PR$, then equation to the line PR is

(a) $89x + 52y + 519 = 0$

(b) $52x + 89y + 519 = 0$

(c) $62x + 89y + 519 = 0$

(d) None of the above

10. A circle is inscribed in an equilateral triangle of side k. The area of any square inscribed in the circle is

(a) k^2 (b) $\frac{k^2}{3}$ (c) $\frac{k^2}{6}$ (d) $6k^2$

11. If ω is an imaginary cube root of unity, then $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & -1-\omega \\ 1 & \omega^5 & \omega^4 \end{vmatrix}$ is equal to

(a) $3\omega^2$ (b) 3ω

(c) $3\omega(\omega^4 - \omega^3)$ (d) $3\omega^4$

12. $\lim_{x\to 5}\frac{\sqrt{1-\cos(x^2-14x+45)}}{(x-5)}$ is equal to

(a) 2 (b) 3 (c) $2^{3/2}$ (d) $-(2)^{3/2}$

13. The area of the region bounded by the curves $x^2 + y^2 = 4$, $x^2 = -\sqrt{2}y$ and $x = y$ is

(a) $\pi + \frac{1}{4}$ (b) $1 + \frac{\pi}{4}$ (c) $\pi + \frac{1}{3}$ (d) π

14. $\int \frac{\tan x}{\sin^2 x \cdot \tan^2 x + \cos^2 x}\, dx$ is equal to

(a) $\frac{1}{2}\tan^{-1}(\cot^2 x) + C$ (b) $\frac{1}{2}\tan^{-1}(\tan^2 2x) + C$

(c) $\tan^{-1}(\tan^2 x) + C$ (d) $\frac{1}{2}\tan^{-1}(\tan^2 x) + C$

15. An infinite GP has first term t and sum 7, then t belongs to

(a) $t > 14$ (b) $0 < t < 14$

(c) $t < -14$ (d) $-14 < t < 0$

16. In a multiple choice question, there are four alternative answers, of which one or more are correct. A candidate will get marks in the question only if he ticks all the correct answers. The candidate decides to tick answers at random. If he is allowed upto three chances to answer the question, then probability that he will get marks in question, is

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$ (c) $\frac{3}{5}$ (d) $\frac{4}{5}$

17. If $P = \begin{bmatrix} -1 & \frac{5}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix}$, then $I + P + P^2 + \ldots \infty$ is equal to

(a) $\begin{bmatrix} -1 & 5 \\ -3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -5 \\ 3 & -4 \end{bmatrix}$

(c) $\frac{-2}{11}\begin{bmatrix} -1 & 5 \\ -3 & 4 \end{bmatrix}$ (d) $\frac{2}{11}\begin{bmatrix} -1 & 5 \\ -3 & 4 \end{bmatrix}$

18. The coefficient of x^7 in the expansion of $(x^3 - x^2 - x + 1)^6$ is

(a) 144 (b) -154 (c) -144 (d) 134

19. The vector $\mathbf{Q} = 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ is to be written as the sum of a vector $\mathbf{Q}_1$ parallel to $\mathbf{P} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and a vector $\mathbf{Q}_2$ perpendicular to $\mathbf{P}$. Then, $\mathbf{Q}_1$ is

(a) $\frac{5}{2}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$ (b) $\frac{3}{2}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$

(c) $\frac{5}{3}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$ (d) $3(\hat{\mathbf{i}} + \hat{\mathbf{j}})$

20. The differential equation which represents the family of curves $y = e^{ax}$ is

(a) $xy' - \log y = 0$ (b) $x \log y = yy'$

(c) $y \log y = xy'$ (d) $y' = ay$

21. If $1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ are the nth roots of unity, then $(5 - \alpha_1)(5 - \alpha_2) \ldots (5 - \alpha_{n-1})$ is equal to

(a) n (b) $5^n + 1$ (c) 5^n (d) $5^n - 1$

22. If $R = \{(5, 5), (7, 7), (11, 11), (15, 15), (7, 15), (5, 11), (5, 15), (5, 7)\}$ is a relation on the set $A = \{5, 7, 11, 15\}$. Then R is

(a) symmetric, transitive but not reflexive

(b) reflexive, transitive but not symmetric

(c) reflexive, symmetric but not transitive

(d) an equivalence relation

23. If minimum value of $f(x) = x^2 + 2bx + 2c^2$ is greater than maximum value of $g(x) = -x^2 - 2cx + b^2$, then for x is real

(a) $|c| \sqrt{2} > b$ (b) $|c| > b\sqrt{2}$

(c) $0 < c < \sqrt{2}b$ (d) no real value of a

24. The left hand derivative of $f(x) = [x]\sin(\pi x)$ at $x = t$, where t is an integer, is

(a) $(t-1)(-1)^t \pi$ (b) $(-1)^{t-1}\ \pi$

(c) $(-1)^t t\ \pi$ (d) $(-1)^{t-1}(t-1)\ \pi$

25. A line with direction cosines proportional to 2, 1, 2 meets each of the line $x = y + k = z$ and $x + k = 2y = 2z$. The coordinates of each of the points of intersection are given by

(a) $(2k, 3k, 3k), (2k, k, k)$ (b) $(3k, 3k, 3k), (k, k, k)$

(c) $(3k, 2k, 3k), (k, k, k)$ (d) $(3k, 2k, 3k), (k, k, 2k)$

26. From the following options, identify the false statement.

(a) $\sim[p \wedge (\sim p)]$ is tautology

(b) $\sim(p \vee q) \equiv (\sim p) \vee (\sim q)$

(c) $\sim[p \vee (\sim q)] \equiv (\sim p) \wedge q$

(d) $[p \wedge q] \wedge (\sim p)$ is a contradiction

27. If the difference between the roots of $x^2 + px - q = 0$ is equal to the difference between the roots of $x^2 - rx + s = 0$, then $p^2 - r^2$ in terms of q and s is

(a) $4(q + s)$ (b) $4(q - s)$

(c) $-4(q + s)$ (d) $4(q - s)$

28. A committee of 4 delegates is to be formed from 2 delegates from Russia, 2 delegates from USA and 4 delegates from India such that it includes atleast 1 delegate from Russia, atleast 1 delegate from USA and atmost 2 delegates from India. Then, the total number of ways in which this committee can be formed is

(a) 40 (b) 31 (c) 51 (d) 41

29. If the mean and standard deviation of 10 observations $x_1, x_2, \ldots, x_{10}$ are 2 and 3 respectively, then the mean of $(x_1 + 1)^2, (x_2 + 1)^2, \ldots, (x_{10} + 1)^2$ is equal to

(a) 18 (b) 13.5

(c) 14.4 (d) 16

30. A plane is parallel to two lines whose direction ratios are $(1, 0, -1)$ and $(-1, 1, 0)$ and it contains the point $(1, 1, 1)$. If it cuts coordinate axes at P, Q, R, then volume of the tetrahedron $OPQR$ is

(a) $\frac{9}{4}$ cu units (b) 9 cu units

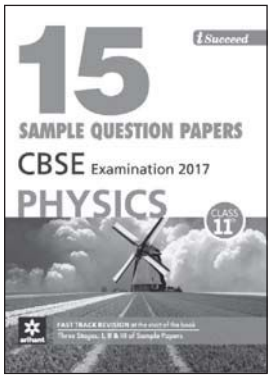
(c) $\frac{9}{5}$ cu units (d) $\frac{9}{2}$ cu units

Answers

1. (b)	**2.** (b)	**3.** (c)	**4.** (d)	**5.** (a)
6. (d)	**7.** (b)	**8.** (c)	**9.** (b)	**10.** (c)
11. (c)	**12.** (d)	**13.** (c)	**14.** (d)	**15.** (b)
16. (a)	**17.** (d)	**18.** (c)	**19.** (b)	**20.** (c)
21. (d)	**22.** (c)	**23.** (b)	**24.** (a)	**25.** (c)
26. (b)	**27.** (c)	**28.** (d)	**29.** (a)	**30.** (d)

15 SAMPLE QUESTION PAPERS

CBSE CLASS XI & XII
EXAMINATION 2017



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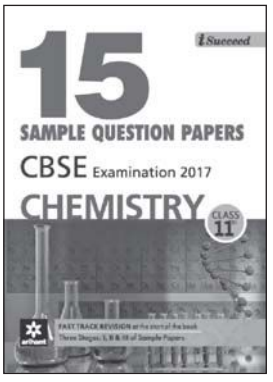

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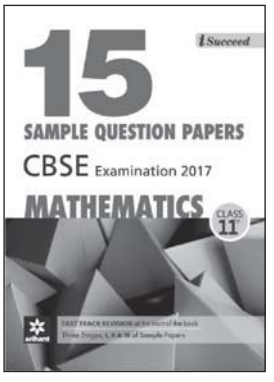

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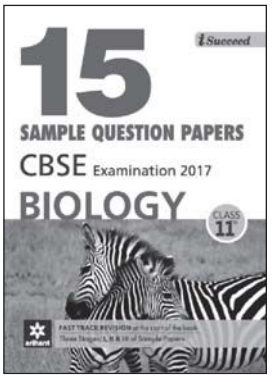

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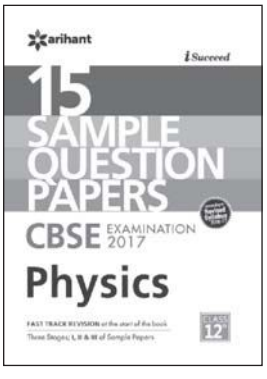

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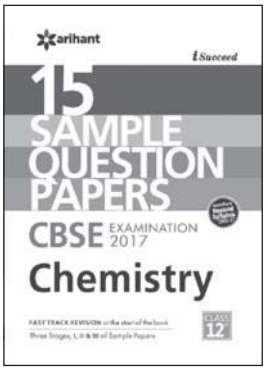

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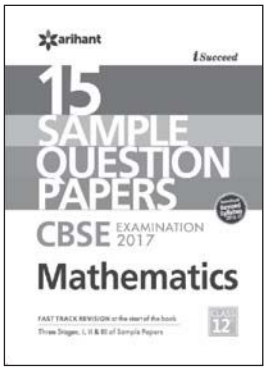

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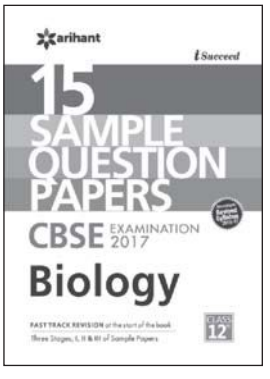

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COMPLEX NUMBERS

A collection of chapterwise Statement type problems for JEE Main

Directions (Q. Nos. 1-30) Each question has four choices (a), (b), (c) and (d), out of which only one is correct. Each question contains Statement I and Statement II.

(a) If both statements are true and Statement II is the correct explanation of Statement I

(b) If both statements are true but Statement II is not the correct explanation of Statement I

(c) If Statement I is true and Statement II is false

(d) If Statement I is false and Statement II is true

1. **Statement I** Let z_1 and z_2 be two complex numbers such that $|z_1 - z_2| = |z_1 + z_2|$, then the orthocentre of ΔAOB is $(z_1 + z_2)/2$ (where O is origin).

Statement II In case of a right angled triangle, orthocentre is that point at which the triangle is right angled.

2. Let the fourth roots of unity be z_1, z_2, z_3 and z_4, respectively.

Statement I $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$

Statement II $z_1 + z_2 + z_3 + z_4 = 0$

3. **Statement I** If n is an odd integer greater than 3 but not a multiple of 3, then $(x+1)^n - x^n - 1$ is divisible by $x^3 + x^2 + x$.

Statement II If n is an odd integer greater than 3 but not a multiple of 3, we have $1 + \omega^n + \omega^{2n} = 3$.

4. **Statement I** $|z_1 - a| < a, |z_2 - b| < b, |z_3 - c| < c$, where a, b and c are positive real numbers, then $|z_1 + z_2 + z_3|$ is greater than $2|a + b + c|$.

Statement II $|z_1 \pm z_2| \le |z_1| \pm |z_2|$.

5. **Statement I** If $z_1 + z_2 = a$ and $z_1 z_2 = b$, where $a = \bar{a}$ and $b = \bar{b}$, then $\arg(z_1 z_2) = 0$.

Statement II The sum and product of two complex numbers are real if and only if they are conjugate of each other.

6. **Statement I** If $\arg(z_1 z_2) = 2\pi$, then both z_1 and z_2 are purely real (z_1 and z_2 have principal arguments).

Statement II Principal argument of complex number lies in $(-\pi, \pi)$.

7. If $z_1 \neq -z_2$ and $|z_1 + z_2| = \left|\left(\frac{1}{z_1}\right) + \left(\frac{1}{z_2}\right)\right|$, then

Statement I $z_1 z_2$ is unimodular.

Statement II Both z_1 and z_2 are unimodular.

8. **Statement I** If z_1, z_2 are the roots of the quadratic equation $az^2 + bz + c = 0$, such that $\text{Im}(z_1 z_2) \neq 0$, then atleast one of a, b and c is imaginary.

Statement II If quadratic equation having real coefficients has complex roots, then roots are always conjugate to each other.

9. **Statement I** The product of all values of $(\cos\alpha + i\sin\alpha)^{3/5}$ is $\cos 3\alpha + i \sin 3\alpha$.

Statement II The product of fifth roots of units is 1.

10. **Statement I** Let z be a complex number, then the equation $z^4 + z + 2 = 0$ cannot have a root, such that $|z| < 1$.

Statement II $|z_1 + z_2| \le |z_1| + |z_2|$

11. **Statement I** If $x + \frac{1}{x} = 1$ and $p = x^{4000} + \frac{1}{x^{4000}}$ and q is the digit at unit place in the number $2^{2^n} + 1$, $n \in N$ and $n > 1$, then the value of $p + q = 8$.

Statement II If ω, ω^2 are the roots of $x + \frac{1}{x} = -1$, then $x^2 + \frac{1}{x^2} = -1$, $x^3 + \frac{1}{x^3} = 2$.

12. **Statement I** Locus of z satisfying the equation $|z-1| + |z-8| = 5$ is an ellipse.

Statement II Sum of focal distances of any point on ellipse is constant.

13. **Statement I** If the equation $ax^2 + bx + c = 0$, $0 < a < b < c$ has non-real complex roots z_1 and z_2, then $|z_1| > 1, |z_2| > 1$.

Statement II Complex roots always occur in conjugate pairs.

14. **Statement I** If $|z_1| = |z_2| = |z_3|, z_1 + z_2 + z_3 = 0$ and $A(z_1), B(z_2), C(z_3)$ are the vertices of ΔABC, then one of the values of $\arg\left(\frac{z_2 + z_3 - 2z_1}{z_3 - z_2}\right)$ is $\frac{\pi}{2}$.

Statement II In equilateral triangle, orthocentre coincides with centroid.

15. **Statement I** If $z_1, z_2, z_3, \ldots, z_n$ are unimodular complex numbers, then

$$|z_1 + z_2 + \ldots + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n}\right|$$

Statement II For any complex number z, $z\bar{z} = |z|^2$.

16. **Statement I** If z is a complex number satisfying $(z-1)^n = z^n$, $n \in N$, then the locus of z is a straight line parallel to imaginary axis.

Statement II The locus of a point equidistant from two given points is the perpendicular bisector of the line segment joining them.

17. Let z_0 be the circumcentre of an equilateral triangle whose affixes are z_1, z_2, z_3.

Statement I $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$

Statement II $z_1^2 + z_2^2 + z_3^2 = 2(z_1z_2 + z_2z_3 + z_3z_1)$

18. **Statement I** The locus of point z satisfying $\left|\frac{3z + i}{2z + 3 + 4i}\right| = \frac{3}{2}$ is a straight line.

Statement II The locus of a point equidistant from two fixed points is a straight line representing the perpendicular bisector of the segment joining the given points.

19. Let z and w be complex numbers such that $|z| = |w|$ and arg (z) denotes the principal argument of z.

Statement I If arg (z) + arg $(w) = \pi$, then $z = -\overline{w}$

Statement II $|z| = |w|$ implies arg (z) – arg $(\overline{w}) = \pi$

20. **Statement I** For any two complex numbers z_1 and z_2, $|z_1 + z_2|^2 \le (1 + \lambda)|z_1|^2 + \left(1 + \frac{1}{\lambda}\right)|z_2|^2$, where λ is a positive real number.

Statement II AM $\ge$ GM

21. **Statement I** For any two complex numbers z_1 and z_2,

$$\left|z_1 + \sqrt{z_1^2 - z_2^2}\right| + \left|z_1 - \sqrt{z_1^2 - z_2^2}\right| = |z_1 + z_2|\,|z_1 - z_2|.$$

Statement II For any two complex numbers z_1 and z_2, $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$.

22. **Statement I** If z_1 and z_2 are two complex numbers such that $|z_1| \le 1, |z_2| \le 1$, then $|z_1 - z_2|^2 \le (|z_1| - |z_2|)^2 + \{\arg(z_1) - \arg(z_2)\}^2$

Statement II $\sin\theta > \theta$ for all $\theta > 0$

23. Let z be a unimodular complex number.

Statement I arg $(z^2 + \bar{z})$ = arg (z)

Statement II $\bar{z} = \cos(\arg z) - i \sin(\arg z)$

24. **Statement I** If a, b, c are distinct real numbers and ω $(\ne 1)$ is a cube root of unity, then $\left|\frac{a + b\omega + c\omega^2}{a\omega^2 + b + c\omega}\right| = 1$

Statement II For any non-zero complex number z, $\left|\frac{z}{\bar{z}}\right| = 1$.

25. Let z_1 and z_2 be the roots of the equation $z^2 + pz + q = 0$. Suppose z_1 and z_2 are represented by points A and B in the argand plane such that $\angle AOB = \alpha$, where O is the origin.

Statement I If $OA = OB$, then $p^2 = 4q\cos^2\frac{\alpha}{2}$.

Statement II If affix of a point P in the argand plane is z, then $ze^{i\alpha}$ is represented by a point Q such that $\angle POQ = \alpha$ and $OP = OQ$.

26. **Statement I** For any non-zero complex number z, $\left|\frac{z}{|z|} - 1\right| \le \arg(z)$.

Statement II $\sin\theta \le \theta$ for all $\theta \ge 0$.

27. **Statement I** For any complex number z,

$$|\text{Re}(z)| + |\text{Im}(z)| \le |z|.$$

Statement II $|\sin\theta| \le 1$ for all θ.

28. Let $a, b \in c, a \ne 0$. Suppose z_1, z_2, z_3 are the roots of the equation $(z + ab)^3 = a^3$.

Statement I Length of a side of a triangle with vertices z_1, z_2 and z_3 is $\sqrt{3}|a|$.

Statement II Roots of $z^3 = a^3$ lie on a circle of radius $|a|$.

29. Let $z_1 = r_1e^{i\theta}, z_2 = r_2e^{i\phi}$, where $\theta, \phi \in [-\pi, \pi]$

Statement I $|z_1 - z_2|^2 \le (r_1 - r_2)^2 + (\theta - \phi)^2$

Statement II $|\sin\theta| \le |\theta|, \forall\, \theta \in R$

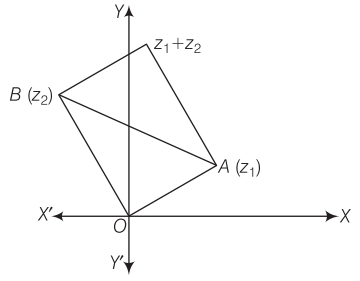
30. Suppose z_1 and z_2 are two distinct non-zero complex numbers.

Statement I $|z_1 - z_2| = \left|\frac{1}{z_1} - \frac{1}{z_2}\right| \Rightarrow |z_1|\,|z_2| = 1$

Statement II If $|z_1 - z_2| = \left|\frac{1}{z_1} - \frac{1}{z_2}\right|$, then both of z_1, z_2 have modulus 10.

Answers with Explanation

1. ***(d)*** From the diagram, when $|z_1 - z_2| = |z_1 + z_2|$, then OAB is right angled triangle. Hence, orthocentre is O.



2. ***(b)*** Fourth roots of unity are $-1, 1, -i$ and i.

$\therefore\ z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$ and $z_1 + z_2 + z_3 + z_4 = 0$

3. ***(c)*** $x^3 + x^2 + x = x(x^2 + x + 1) = x(x - w)(x - w^2)$

Now, $f(x) = (x + 1)^n - x^n - 1$ is divisible by $x^3 + x^2 + x$.

Then, $f(0) = 0, f(w) = 0, f(\omega^2) = 0$.

Now, $f(0) = (0 + 1)^n - 0^n - 1 = 0$

$f(\omega) = (\omega + 1)^n - \omega^n - 1$

$= (-\omega^2)^n - \omega^n - 1 = -(\omega^{2n} + \omega^n + 1) = 0$

[as n is not a multiple of 3]

Similarly, we have $f(\omega^2) = 0$

Hence, Statement I is correct but Statement II is false.

4. ***(d)*** $|z_1 + z_2 + z_3|$

$= |z_1 - a + z_2 - b + z_3 - c + (a + b + c)|$

$\le |z_1 - a| + |z_2 - b| + |z_3 - c| + |a + b + c|$

$\le 2|a + b + c|$

Hence, $|z_1 + z_2 + z_3|$ is less than $2|a + b + c|$.

5. ***(a)*** First, let the two complex numbers be conjugate of each other. Let complex numbers be $z_1 = x + iy$ and $z_2 = x - iy$.

Then, $z_1 z_2 = (x + iy)(x - iy) = x^2 - i^2 y^2 = x^2 + y^2$, which is real.

Conversely, let z_1 and z_2 be two complex numbers such that their sum $z_1 + z_2$ and product $z_1 z_2$ both are real.

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Then, $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

and $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

Now, $z_1 + z_2$ and $z_1 z_2$ are real.

Hence, $b_1 + b_2 = 0$ and $a_1 b_2 + a_2 b_1 = 0$

[$\because z$ is real $\Rightarrow \text{Im}(z) = 0$]

$\Rightarrow\ b_2 = -b_1$ and $a_1 b_2 + a_2 b_1 = 0$

$= -b_1$ and $-a_1 b_1 + a_2 b_1 = 0$

$= -b_1$ and $(a_2 - a_1) b_1 = 0$

$= -b_1$ and $a_2 - a_1 = 0 = -b_1$ and $a_2 = a_1$

$\Rightarrow\ z_2 = a_2 + ib_2 = a_1 - ib_1 = \bar{z}_1$

Hence, z_1 and z_2 are conjugate of each other. Hence, Statement II is true.

Also, in Statement I, $a = \bar{a}$ and $b = \bar{b}$, then a and b are real. Thus, $z_1 + z_2$ and $z_1 z_2$ are real. So, $z_2 = \bar{z}_2$

$\Rightarrow \arg(z_1 z_2) = \arg(z_1 \bar{z}_1) = \arg(|z_1|^2) = 0$

Hence, Statements I and II are correct and Statement II is the correct explanation of Statement I.

6. ***(a)*** $\arg(z_1 z_2) = 2\pi \Rightarrow \arg(z_1) + \arg(z_2) = 2\pi$

$\Rightarrow\ \arg(z_1) = \arg(z_2) = \pi$

As principal arguments are from $-\pi$ to π.

Hence, both the complex numbers are purely real. Hence, both the statements are true and Statement II is the correct explanation of Statement I.

7. ***(c)*** $|z_1 + z_2| = \left|\dfrac{z_1 + z_2}{z_1 z_2}\right| \Rightarrow |z_1 + z_2|\left(1 - \dfrac{1}{|z_1 z_2|}\right) = 0$

$\Rightarrow\ |z_1 z_2| = 1$

Hence, Statement I is true. However, it is not necessary that $|z_1| = |z_2| = 1$. Hence Statement II is false.

8. ***(a)*** We have, $az^2 + bz + c = 0$ ...(i)

and z_1, z_2 [roots of Eq. (i)] are such that $\text{Im}(z_1 z_2) \neq 0$. So, z_1 and z_2 are not conjugate of each other, i.e. complex roots of Eq. (i) are not conjugate of each other which implies that coefficients a, b, c cannot all be real. Hence, atleast one of a, b, c is imaginary.

9. ***(b)*** Let $x = (\cos\alpha + i\sin\alpha)^{3/5}$

$\Rightarrow\ x^5 = (\cos\alpha + i\sin\alpha)^3$

$\Rightarrow\ x^5 - (\cos 3\alpha + i\sin 3\alpha) = 0$

$\Rightarrow$ Product of roots $= \cos 3\alpha + i\sin 3\alpha$

Also, product of roots of the equation $x^5 - 1 = 0$ is 1.

Hence, Statement II is true. But it is not the correct explanation of Statement I.

10. ***(a)*** Suppose there exists a complex number z which satisfies the given equation and is such that $|z| < 1$.

Then, $z^4 + z + 2 = 0 \Rightarrow -2 = z^4 + z$

$\Rightarrow\ |-2| = |z^4 + z|$

$\Rightarrow\ 2 \le |z^4| + |z| \Rightarrow 2 < 2$, because $|z| < 1$

But $z < 2$ is not possible.

Hence, given equation cannot have root z such that $|z| < 1$.

11. ***(d)*** $x + \dfrac{1}{x} = 1 \ \Rightarrow\ x^2 - x + 1 = 0$

$\therefore\ x = -\omega, -\omega^2$

Now for $x = -\omega$, $P = \omega^{4000} + \dfrac{1}{\omega^{4000}} = \omega + \dfrac{1}{\omega} = -1$

Similarly, for $x = -\omega^2$, $p = -1$, for $n > 1$, $2^n = 4K$

$\therefore\ 2^{2n} = 2^{4K} = (16)^K$ = a number with last digit 6

$\Rightarrow\ q = 6 + 1 = 7$

Hence, $p + q = -1 + 7 = 6$

12. ***(d)*** Statement II is true as it is the definition of an ellipse. Statement I is false as distance between 1 and 8 is 7 but $|z - 1| + |z - 8| = 5 < 7$.

Hence, no such z exists.

13. ***(a)*** If roots of $ax^2 + bx + c = 0, 0 < a < b < c$, are non-real, then they will be the conjugate of each other. Hence, $z_2 = \bar{z}_1 \Rightarrow |z_1| = |z_2|$

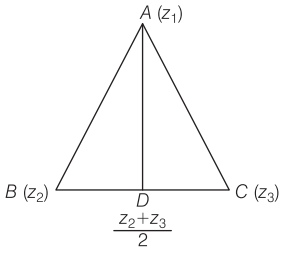
Now, $z_1 z_2 = \dfrac{c}{a} > 1 \Rightarrow |z_1|^2 > 1$

$\Rightarrow\ |z_1| > 1 \ \Rightarrow\ |z_2| > 1$

14. ***(b)*** Since $|z_1| = |z_2| = |z_3|$, circumcentre of triangle is origin.

Also, $\frac{z_1 + z_2 + z_3}{3} = 0$

Centroid coincide with circumcentre. Hence, triangle is equilateral.



$$\arg\left(\frac{z_2 + z_3 - 2z_1}{z_3 - z_2}\right) = \arg\left(2\left(\frac{\frac{z_2 + z_3}{2} - z_1}{z_3 - z_2}\right)\right)$$

$$= \arg\left(\frac{\frac{z_2 + z_3}{2} - z_1}{z_3 - z_2}\right)$$

$(z_2 + z_3)/2$ is the mid-point of side BC. Clearly, line joining A and mid-point of BC will be perpendicular to side BC.

Thus, $$\arg\left(\frac{\frac{z_2 + z_3}{2} - z_1}{z_3 - z_2}\right) = \frac{\pi}{2}$$

Hence, Statement II is also true. However, it does not explain Statement I.

15. ***(a)*** We have, $|z_1| = |z_2| = ... = |z_n| = 1$

$$\Rightarrow \quad |z_1|^2 = |z_2|^2 = ... = |z_n|^2 = 1$$

$$\Rightarrow \quad z_1\bar{z}_1 = z_2\bar{z}_2 = ... = z_n\bar{z}_n = 1$$

$$\Rightarrow \quad \bar{z}_1 = \frac{1}{z_1}, \bar{z}_2 = \frac{1}{z_2}, ..., \bar{z}_n = \frac{1}{z_n}$$

Now, $|z_1 + z_2 + ... + z_n|$

$$= |\overline{z_1 + z_2 + ... + z_n}|$$

$$= |\bar{z}_1 + \bar{z}_2 + ... + \bar{z}_n| \qquad [\because |z| = |\bar{z}|]$$

$$= \left|\frac{1}{z_1} + \frac{1}{z_2} + ... + \frac{1}{z_n}\right|$$

16. ***(a)*** Clearly, Statement II is true.

Now, $$(z-1)^n = z^n \Rightarrow \left(\frac{z-1}{z}\right)^n = 1$$

$$\Rightarrow \quad \frac{z-1}{z} = 1^{1/n}$$

$$\Rightarrow \quad \frac{z-1}{z} = e^{i2r\pi/n}, r = 0, 1, 2, ..., n-1$$

$$\Rightarrow \quad \left|\frac{z-1}{z}\right| = 1 \Rightarrow |z - (1 + 0i)| = |z - 0|$$

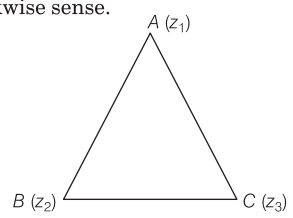
$\Rightarrow z$ is equidistant from $(0, 0)$ and $(1, 0)$.

$\Rightarrow$ Locus of z is a straight line parallel to imaginary axis.

Hence, Statements I and II are true and Statement II is the correct explanation of Statement I.

17. ***(c)*** Let ABC be the given equilateral triangle such that the affixes of A, B, C are z_1, z_2 and z_3 respectively.

Clearly, **AC** is obtained by rotating **AB** through $\frac{\pi}{3}$ in anti-clockwise sense.



$$\therefore \quad \mathbf{AC} = \mathbf{AB}\, e^{i\pi/3}$$

$$\Rightarrow \quad z_3 - z_1 = (z_2 - z_1)\, e^{i\pi/3} \qquad ...(i)$$

Similarly, we have

$$\mathbf{BA} = \mathbf{BC}\, e^{i\pi/3}$$

$$\Rightarrow \quad z_1 - z_2 = (z_3 - z_2)\, e^{i\pi/3} \qquad ...(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{z_3 - z_1}{z_1 - z_2} = \frac{z_2 - z_1}{z_3 - z_2}$$

$$\Rightarrow \quad z_3^2 - z_1z_3 - z_2z_3 + z_1z_2 = z_1z_2 - z_1^2 - z_2^2 + z_1z_2$$

$$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1 \qquad ...(iii)$$

So, Statement II is false.

Since, circumcentre and centroid of an equilateral triangle coincide.

$$\therefore \quad z_0 = \frac{z_1 + z_2 + z_3}{3}$$

$$\Rightarrow \quad (z_1 + z_2 + z_3)^2 = 9z_0^2$$

$$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 + 2(z_1z_2 + z_2z_3 + z_3z_1) = 9z_0^2$$

$$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = 3z_0^2 \qquad [\text{using Eq. (iii)}]$$

So, Statement I is true.

18. ***(a)*** Clearly, Statement II is true.

We have, $$\left|\frac{3z + i}{2z + 3 + 4i}\right| = \frac{3}{2}$$

$$\Rightarrow \quad \left|z + \frac{1}{3}i\right| = \left|z + \frac{3}{2} + 2i\right|$$

$$\Rightarrow \quad \left|z - \left(0 - \frac{1}{3}i\right)\right| = \left|z - \left(-\frac{3}{2} - 2i\right)\right|$$

$\Rightarrow z$ is equidistant from $\left(0, -\frac{1}{3}\right)$ and $\left(-\frac{3}{2}, -2\right)$.

$\Rightarrow$ Locus of z is a straight line. [using Statement II]

19. ***(c)*** Let $z = r(\cos\theta + i\sin\theta)$ and $w = r(\cos\alpha + i\sin\alpha)$

Then, $\arg(z) + \arg(w) = \pi$

$$\Rightarrow \quad \theta + \alpha = \pi$$

$$\Rightarrow \quad \theta = \pi - \alpha$$

$$\therefore \quad z = r(-\cos\alpha + i\sin\alpha)$$

$$= -r(\cos\alpha - i\sin\alpha) = -\bar{w}$$

So, Statement I is true.

Clearly, $|z| = |w|$ need not imply that $\arg(z) - \arg(\bar{w}) = \pi$, because if we take $z = i$ and $w = -i$, then $|z| = |w|$.

But $\arg(z) - \arg(\bar{w}) = \arg(i) - \arg(i) = 0 \neq \pi$

Hence, Statement I is true and Statement II is false.

20. ***(a)*** We have, $(z_1 + z_2)^2 = |z_1|^2 + |z_2| + 2\text{Re}\,(z_1\bar{z}_2)$

$\Rightarrow \quad |z_1 + z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2| \quad [\because \text{Re}\,(z) \le |z|]$

$\Rightarrow \quad (z_1 + z_2)^2 \le |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \quad \ldots(i)$

Now, $\text{AM} \ge \text{GM}$

$$\Rightarrow \quad \frac{\lambda|z_1|^2 + \frac{1}{\lambda}|z_2|^2}{2} \ge \sqrt{\lambda|z_1|^2 \times \frac{1}{\lambda}|z_2|^2}$$

$$\Rightarrow \quad \lambda|z_1|^2 + \frac{1}{\lambda}|z_2|^2 \ge 2|z_1||z_2|$$

$$\Rightarrow \quad 2|z_1||z_2| \le \lambda|z_1|^2 + \frac{1}{\lambda}|z_2|^2$$

$$\Rightarrow |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \le |z_1|^2 + |z_2|^2 + \lambda|z_1|^2 + \frac{1}{\lambda}|z_2|^2$$

$$\Rightarrow \quad (|z_1| + |z_2|)^2 \le |z_1|^2(1 + \lambda) + |z_2|^2\left(1 + \frac{1}{\lambda}\right)$$

$$\Rightarrow \quad |z_1 + z_2|^2 \le (1 + \lambda)|z_1|^2 + \left(1 + \frac{1}{\lambda}\right)|z_2|^2$$

Hence, Statements I and II are true and Statement II is the correct explanation of Statement I.

21. ***(a)*** We have, $|z_1 + z_2|^2 + |z_1 - z_2|^2$

$= |z_1|^2 + |z_2|^2 + 2\text{Re}\,(z_1\bar{z}_2) + |z_1|^2 + |z_2|^2 - 2\text{Re}\,(z_1\bar{z}_2)$

$= 2(|z_1|^2 + |z_2|^2)$

So, Statement II is true.

Also, $\{|z_1 + \sqrt{z_1^2 - z_2^2}| + |z_1 - \sqrt{z_1^2 - z_2^2}|\}^2$

$= |z_1 + \sqrt{z_1^2 - z_2^2}|^2 + |z_1 - \sqrt{z_1^2 - z_2^2}|^2 + 2|z_1 + \sqrt{z_1^2 - z_2^2}||z_1 - \sqrt{z_1^2 - z_2^2}|$

$= 2\{|z_1|^2 + |\sqrt{z_1^2 - z_2^2}|^2\} + 2|z_1^2 - (z_1^2 - z_2^2)|$ [using Statement II]

$= 2\{|z_1|^2 + |z_1^2 - z_2^2|\} + 2|z_2|^2$

$= 2(|z_1|^2 + |z_2|^2) + 2|z_1 + z_2||z_1 - z_2|$

$= |z_1 + z_2|^2 + |z_1 - z_2|^2 + 2|z_1 + z_2||z_1 - z_2|$ [using Statement II]

$= (|z_1 + z_2| + |z_1 - z_2|)^2$

$\therefore \quad |z_1 + \sqrt{z_1^2 - z_2^2}| + |z_1 - \sqrt{z_1^2 - z_2^2}|$

$= |z_1 + z_2| + |z_1 - z_2|$

So, Statement I is also true. Also, Statement II is a correct explanation for Statement I.

22. ***(c)*** Let $|z_1| = r_1$, $|z_2| = r_2$, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$.

Then, $|z_1 - z_2|^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$

$\Rightarrow |z_1 - z_2|^2 = (r_1^2 + r_2^2 - 2r_1r_2) + 2r_1r_2$ $[1 - \cos(\theta_1 - \theta_2)]$

$$\Rightarrow |z_1 - z_2|^2 = (r_1 - r_2)^2 + 4r_1r_2\sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)$$

$$\Rightarrow |z_1 - z_2|^2 \le (r_1 - r_2)^2 + 4r_1r_2\left(\frac{\theta_1 - \theta_2}{2}\right)^2 \quad [\because \sin\theta < \theta]$$

$\Rightarrow |z_1 - z_2|^2 \le (r_1 - r_2)^2 + (\theta_1 - \theta_2)^2 \quad [\because r_1 \le 1, r_2 \le 1]$

$\therefore \quad |z_1 - z_2|^2 \le (|z_1| - |z_2|)^2 + [\arg(z_1) - \arg(z_2)]^2$

Hence, Statement I is true.

We know that $\sin\theta < \theta$ for all $\theta > 0$.

So, Statement II is not true.

23. ***(d)*** We have, $|z| = 1$

$\therefore \quad z = \cos\theta + i\sin\theta$, where $\theta = \arg(z)$

$\Rightarrow \quad \bar{z} = \cos\theta - i\sin\theta = \cos(\arg z) - i\sin(\arg z)$

So, Statement II is true.

Now, $\arg(z^2 + \bar{z}) = \arg(\cos 2\theta + i\sin 2\theta + \cos\theta - i\sin\theta)$

$$= \tan^{-1}\left(\frac{\sin 2\theta - \sin\theta}{\cos 2\theta + \cos\theta}\right)$$

$$= \tan^{-1}\left(\frac{\tan\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\arg(z)$$

So, Statement I is false.

24. ***(b)*** We have,

$$a\omega^2 + b + c\omega = \frac{1}{\omega}(a\omega^3 + b\omega + c\omega^2) = \frac{1}{\omega}(a + b\omega + c\omega^2)$$

$\Rightarrow a + b\omega + c\omega^2 = \omega(a\omega^2 + b + c\omega)$

$$\Rightarrow \left|\frac{a + b\omega + c\omega^2}{a\omega^2 + b + c\omega}\right| = |\omega| = 1$$

So, Statement I is true.

For any non-zero complex number z, we have

$$\left|\frac{z}{\bar{z}}\right| = \frac{|z|}{|\bar{z}|} = 1 \qquad [\because |z| = |\bar{z}|]$$

So, Statement II is true. But it is not a correct explanation for Statement I.

25. ***(a)*** Clearly, Statement II is true.

We have, $z_1 + z_2 = -p$ and $z_1z_2 = q$

It is given that $OA = OB$ and $\angle AOB = \alpha$

$$\therefore \quad z_2 = z_1 e^{i\alpha} \quad \Rightarrow \quad \frac{z_2}{z_1} = e^{i\alpha}$$

$$\Rightarrow \quad \frac{z_2}{z_1} + 1 = \cos\alpha + i\sin\alpha + 1$$

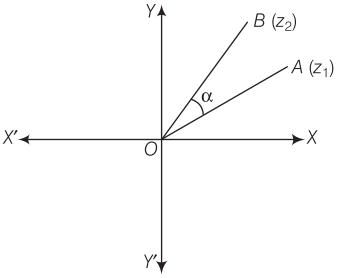
$$\Rightarrow \quad \frac{z_2 + z_1}{z_1} = 2\cos\frac{\alpha}{2}e^{i\alpha/2}$$

$$\Rightarrow \quad \frac{-p}{z_1} = 2\cos\frac{\alpha}{2}e^{i\alpha/2}$$

$$\Rightarrow \quad p^2 = 4z_1^2\cos^2\frac{\alpha}{2}e^{i\alpha}$$

$$\Rightarrow \quad p^2 = 4z_1z_2\cos^2\frac{\alpha}{2} \Rightarrow p^2 = 4q\cos^2\frac{\alpha}{2}$$

Hence, Statement I is true.



26. ***(a)*** Let $z = r(\cos\theta + i\sin\theta)$. Then, $|z| = r$ and $\arg(z) = \theta$

$$\therefore \quad \left|\frac{z}{|z|} - 1\right| = |\cos\theta + i\sin\theta - 1|$$

$$\Rightarrow \quad \left|\frac{z}{|z|} - 1\right| = \left|-2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right|$$

$\Rightarrow \quad \left|\frac{z}{|z|}-1\right| = 2\left|\sin\frac{\theta}{2}\right| \le \theta \qquad \left[\because \sin\frac{\theta}{2} \le \frac{\theta}{2}\right]$

$\Rightarrow \quad \left|\frac{z}{|z|-1}\right| \le \arg(z)$

Hence, Statements I and II are true and Statement II is the correct explanation of Statement I.

27. ***(d)*** Let $z = r(\cos\theta + i\sin\theta)$. Then, $|z| = r$ and $\arg(z) = \theta$

Now, $|\text{Re}(z)| + |\text{Im}(z)| = |r\cos\theta| + |r\sin\theta|$

$|\text{Re}(z)| + |\text{Im}(z)| = r(|\cos\theta| + |\sin\theta|)$

$\Rightarrow \quad \{|\text{Re}(z)| + |\text{Im}(z)|\}^2 = r^2(|\cos\theta| + |\sin\theta|)^2$

$\Rightarrow \quad \{|\text{Re}(z)| + |\text{Im}(z)|\}^2 = r^2(1 + |2\sin\theta\cos\theta|)$

$\Rightarrow \quad \{|\text{Re}(z)| + |\text{Im}(z)|\}^2 = r^2(1 + |\sin 2\theta|)$

$\Rightarrow \quad \{|\text{Re}(z) + |\text{Im}(z)|\}^2 \le 2r^2$

$\Rightarrow \quad |\text{Re}(z)| + |\text{Im}(z)| \le \sqrt{2}|z|$

Hence, Statement I is not true. But Statement II is true.

28. ***(b)*** Statement II is true, as $z^3 = a^3$

$\Rightarrow \quad |z|^3 = |a|^3 \quad \Rightarrow \quad |z| = |a| \quad (z + ab)^3 = a^3$

$\Rightarrow \quad z + ab = a, a\omega, a\omega^2$

Let $z_1 = a - ab, z_2 = a\omega - ab$ and $z_3 = a\omega^2 - ab$

We have, $|z_2 - z_1| = |a||\omega - 1| = \sqrt{3}a$

$|z_3 - z_2| = \sqrt{3}a, |z_1 - z_3| = \sqrt{3}a$

Thus, Statement I is also true but Statement II is not a correct explanation of Statement I.

29. ***(a)*** We know that $|\sin\theta| \le \theta, \forall\, \theta \ge 0$

If $\theta < 0$, then $|\sin(-\theta)| \le -\theta \Rightarrow |\sin\theta| \le |\theta|$

Thus, $|\sin\theta| \le |\theta| \ \forall\, \theta \in R$ and Statement II is true.

Now, $|z_1 - z_2|^2 = (r_1\cos\theta - r_2\cos\phi)^2 + (r_1\sin\theta - r_2\sin\phi)^2$

$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta - \phi)$

$= (r_1 - r_2)^2 + 4r_1r_2\sin^2\left(\frac{\theta - \phi}{2}\right)$

$\le (r_1 - r_2)^2 + 4(1)(1)\left(\frac{\theta - \phi}{2}\right)^2$

Thus, Statement I is also true and Statement II is a correct explanation for it.

30. ***(c)*** $|z_1 - z_2| = \left|\frac{1}{z_1} - \frac{1}{z_2}\right| = \frac{|z_2 - z_1|}{|z_2 z_1|}$

As $z_1 \ne z_2$, we get $|z_1 z_2| = 1$

$\Rightarrow \quad |z_1||z_2| = 1$

$\therefore$ Statement I is true.

Statement II is false.

For example, if we take $z_1 = 3, z_2 = \frac{1}{3}$

Continued from Page 34

Here, $n = 8$ and $\Sigma(x_i - \bar{x})^2 = 74$

$\therefore \quad \sigma = \sqrt{\frac{1}{n}\Sigma(x_i - \bar{x})^2} = \sqrt{\frac{74}{8}} = \sqrt{9.25} = 3.04$ cm

24. ***(b)*** Since, two lines intersect at a point. Then, shortest distance between them is zero.

$\therefore \begin{vmatrix} k & 2 & 3 \\ 3 & k & 2 \\ 1 & 1 & -2 \end{vmatrix} = 0 \Rightarrow k(-2k-2) - 2(-6-2) + 3(3-k) = 0$

$\Rightarrow 2k^2 + 5k - 25 = 0 \Rightarrow (2k-5)(k+5) = 0 \Rightarrow k = \frac{5}{2}, -5$

Hence, integer value of k is -5.

25. ***(a)*** The equation of the plane through the intersection of given planes is

$(x + y + z - 1) + k(2x + 3y - z + 4) = 0 \qquad \ldots(i)$

$\Rightarrow \quad x(1 + 2k) + y(1 + 3k) + z(1 - k) - 1(1 - 4k) = 0$

$\because$ Plane is parallel to X-axis.

$x(1 + 2k) = 0 \quad \Rightarrow \quad k = -\frac{1}{2}$

Put $k = -\frac{1}{2}$ in Eq. (i), we get the required equation of plane which is

$2(x + y + z - 1) - 2x - 3y + z - 4 = 0 \Rightarrow y - 3z + 6 = 0$

26. ***(d)*** Let $\mathbf{a} = 4\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{b} = 3\hat{\mathbf{i}} + 8\hat{\mathbf{j}} + \hat{\mathbf{k}}$

$\therefore \quad \mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 4 & 6 & -1 \\ 3 & 8 & 1 \end{vmatrix} = 14\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 14\hat{\mathbf{k}}$

$\Rightarrow \quad \hat{\mathbf{c}} = \frac{14\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 14\hat{\mathbf{k}}}{\sqrt{(14)^2 + (7)^2 + (14)^2}} = \frac{14\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 14\hat{\mathbf{k}}}{21}$

$\therefore$ Required vector $= 12 \cdot \frac{(14\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 14\hat{\mathbf{k}})}{21} = 8\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 8\hat{\mathbf{k}}$

27. ***(a)*** Given, $\frac{dr}{dt} = 400\pi$ cm^3/min

$\therefore \quad \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 400\pi \Rightarrow 3r^2\frac{dr}{dt} = \frac{1200\pi}{4\pi}$

$\left(\frac{dr}{dt}\right)_{r=5} = \frac{1200}{4 \times 3 \times 25} = \frac{100}{25} = 4$ cm/min

28. ***(b)*** $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$

$= 2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + 2\cos\frac{4\pi}{13}\cos\frac{\pi}{13}$

$= 2\cos\frac{\pi}{13}\left[\cos\frac{9\pi}{13} + \cos\frac{4\pi}{13}\right] = 2\cos\frac{\pi}{13}\left[2\cos\frac{\pi}{2}\cdot\cos\frac{5\pi}{26}\right] = 0$

29. ***(a)*** $\sin\left[\sin^{-1}\left(\frac{1}{3}\right) + \sec^{-1}(3)\right] + \cos[\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}(2)]$

$= \sin\left[\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}\left(\frac{1}{3}\right)\right] + \cos\left[\tan^{-1}\left(\frac{1}{2}\right) + \cot^{-1}\left(\frac{1}{2}\right)\right]$

$= \sin\frac{\pi}{2} + \cos\frac{\pi}{2}$

$\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} \text{ and } \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$

$= 1$

30. ***(d)*** At $x = 1$, RHD $= \lim_{h \to 0^+}\frac{f(1+h) - f(1)}{h}$

$= \lim_{h \to 0}\frac{2 - (1+h) - (2-1)}{h} = -1$

LHD $= \lim_{h \to 0^-}\frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0}\frac{(1-h) - (2-1)}{-h} = 1$

$\therefore$ LHS $\ne$ RHD

Learn Through Problems

DIFFERENTIAL EQUATION

Section targetted to provide stepwise learning & practice through selective problems

1 The differential equation that represents all parabolas having their axis of symmetry coincident with X-axis, is

(a) $yy_1^2 + y_2 = 0$ (b) $yy_2 + y_1^2 = 0$

(c) $y_1^2 + yy_2 = 0$ (d) $yy_2 + y_1 = 0$

(b) ***Step I*** Write the given equation involving independent variable x (say), dependent variable y (say) and the arbitrary constants.

The equation that represents a family of parabolas having their axis of symmetry coincident with X-axis is

$$y^2 = 4a(x - h) \quad \ldots(i)$$

Step II Obtain the number of arbitrary constants in Step I. Let there be n arbitrary constants.

This equation contains two arbitrary constants.

Step III Differentiate the relation in Step I, n times with respect to x. We will differentiate the Eq. (i) twice.

On differentiating Eq. (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a \Rightarrow y\frac{dy}{dx} = 2a \quad \ldots(ii)$$

Again differentiating Eq. (ii) w.r.t. x, we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \Rightarrow y\,y_2 + (y_1)^2 = 0$$

Step IV Eliminate arbitrary constants with the help of n equations involving differential coefficients obtained in Step III and an equation in Step I. The equation so obtained is the desired differential equation.

The required differential equation is $yy_2 + (y_1)^2 = 0$.

TRY THESE

1. The differential equation of all circles passing through the origin and having their centres on the X-axis, is

(a) $y^2 = x^2 + 2xy\frac{dy}{dx}$ (b) $y^2 = x^2 - 2xy\frac{dy}{dx}$

(c) $x^2 = y^2 + xy\frac{dy}{dx}$ (d) $x^2 = y^2 + 3xy\frac{dy}{dx}$

2. The differential equation of the family of lines through the origin, is

(a) $x\frac{dy}{dx} - y = 0$ (b) $x + \frac{dy}{dx} = 0$ (c) $\frac{dy}{dx} = y$ (d) $\frac{dy}{dx} = x$

3. The differential equation of family of circles with fixed radius 5 units and centre on the line $y = 2$, is

(a) $(y-2)^2 y'^2 = 25 - (y-2)^3$ (b) $(y-2)^2 y'^2 = 25(y-2)^2$

(c) $(x-2)^2 y'^2 = 25 - (x-2)^3$ (d) $(x-2)^2 y'^2 = 25 - (x-2)^2$

1. (a) **2.** (a) **3.** (b)

2 The differential equation $x\frac{dy}{dx} - y = x^3$, has the general solution

(a) $y - x^3 = 2Cx$ (b) $2y - x^3 = Cx$

(c) $2y + x^2 = 2Cx$ (d) $y + x^2 = 2Cx$

(b) ***Step I*** Write the given differential equation in the form of $\frac{dy}{dx} + Py = Q$ or $\frac{dx}{dy} + Rx = S$.

We have, $x\frac{dy}{dx} - y = x^3 \Rightarrow \frac{dy}{dx} - \frac{1}{x}y = x^2$

which is linear differential equating of the form $\frac{dy}{dx} + Py = Q$.

Step II Obtain the integrating factor according to the equation.

(i) If equation is of the form $dy/dx + Py = Q$, then integrating factor (IF) $= e^{\int P\,dx}$

(ii) If equation is of the form $dy/dx + Rx = S$, then integrating factor (IF) $= e^{\int R dy}$

$$\therefore \text{IF} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

Step III Multiply both the sides by IF and integrate to get the required solution.

$$\therefore \frac{1}{x}\cdot\frac{dy}{dx} - \frac{y}{x^2} = x \Rightarrow \frac{x\frac{dy}{dx} - y}{x^2} = x \Rightarrow \frac{d}{dx}\left(\frac{y}{x}\right) = x$$

$$\Rightarrow \frac{y}{x} = \frac{x^2}{2} + C' \Rightarrow y = \frac{x^3}{2} + C'x \Rightarrow 2y = x^3 + 2C'x$$

$$\Rightarrow 2y - x^3 = Cx \quad [\text{where } 2C' = C]$$

TRY THESE

1. If $\frac{dx}{dy} = (e^y - x)$, where $y(0) = 0$, then y is expressed explicitly as

(a) $\frac{1}{2}\ln(1 + x^2)$ (b) $\ln(1 + x^2)$ (c) $\ln\left(x + \sqrt{1 + x^2}\right)$ (d) $\ln\left(x + \sqrt{1 - x^2}\right)$

2. Let $y(x)$ be the solution of the differential equation $(x\log x)\frac{dy}{dx} + y = 2x\log x$, $(x \geq 1)$. Then, $y(e)$ is equal to

(a) 2 (b) 2e (c) e (d) 0

3. Consider, the differential equation $y\,dx - (x + y^2)dy = 0$. If for $y = 1$, x takes value 1, then the value of x when $y = 4$ is

(a) 64 (b) 9 (c) 16 (d) 36

1. (c) **2.** (a) **3.** (c)

3 If a is an arbitrary constant, then solution of the differential equation $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$, is

(a) $y\sqrt{1-y^2}+x\sqrt{1-x^2}=a$

(b) $x\sqrt{1-y^2}-y\sqrt{1-x^2}=a$

(c) $y\sqrt{1-y^2}-x\sqrt{1-x^2}=a$

(d) $x\sqrt{1-y^2}+y\sqrt{1-x^2}=a$

(d) ***Step I*** Write down the given differential equation in the form of $\frac{dy}{dx}=f(x)\,g(y)$

Here, $\frac{dy}{dx}=-\sqrt{\frac{1-y^2}{1-x^2}}$

Step II Separate the functions of x and y and write the equation in the form of $g'(y)\,dy=f'(x)\,dx$.

$$\therefore \quad \frac{1}{\sqrt{1-y^2}}dy=\frac{-1}{\sqrt{1-x^2}}dx$$

Step III Integrate both the sides and simplify it to get the required solution.

$$\therefore \quad \int\frac{1}{\sqrt{1-y^2}}dy=-\int\frac{1}{\sqrt{1-x^2}}dx$$

$$\Rightarrow \quad \sin^{-1}y=-\sin^{-1}x+\sin^{-1}a$$

$$\Rightarrow \quad \sin^{-1}x+\sin^{-1}y=\sin^{-1}a$$

$$\Rightarrow \quad \sin^{-1}[x\sqrt{1-y^2}+y\sqrt{1-x^2}]=\sin^{-1}a$$

$$\Rightarrow \quad x\sqrt{1-y^2}+y\sqrt{1-x^2}=a$$

TRY THESE

1. Tangent is drawn at any point P of a curve which passes through (1, 1) cutting X-axis and Y-axis at A and B, respectively. If $AP:PB=3:1$, then

(a) the differential equation of the curve is $3x\frac{dy}{dx}+y=0$ and the curve passes through (1/8, 2)

(b) the differential equation of the curve is $3x\frac{dy}{dx}-y=0$ and the curve passes through (1/8, −2)

(c) the curve is passing through (−1/8, −2)

(d) the normal at (1, 1) is $x+3y=4$

2. The differential equation $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{y}$ determines a family of circles with

(a) variable radii and fixed centre at (0, 1)

(b) variable radii and a fixed centre at (0, −1)

(c) fixed radius 1 and variable centre along the X-axis

(d) fixed radius 1 and variable centre along the Y-axis

3. The interval contained in the domain of definition of non-zero solutions of the differential equation $(x-3)^2y'+y=0$ is

(a) $(-\pi,\pi)$ (b) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(c) $(0,\pi)$ (d) $(0,2\pi)$

1. (a) **2.** (c) **3.** (b)

4 The general solution of $y^2dx+(x^2-xy+y^2)\,dy=0$, is

(a) $\tan^{-1}\frac{x}{y}+\log y+C=0$

(b) $2\tan^{-1}\frac{x}{y}+\log x+C=0$

(c) $\log(y+\sqrt{x^2+y^2})+\log y+C=0$

(d) $\log y=\tan^{-1}\frac{y}{x}+C$

(d) ***Step I*** Write the given differential equation as

$$\frac{dy}{dx}=\frac{x^nf(y/x)}{x^ng(y/x)}=\frac{f(y/x)}{g(y/x)}=F(y/x)$$

We have, $y^2dx+(x^2-xy+y^2)\,dy=0$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{y^2}{x^2-xy+y^2}\Rightarrow \frac{dy}{dx}=\frac{-(y/x)^2}{1-(y/x)+(y/x)^2} \quad ...(i)$$

Step II Put $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$

On putting $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in Eq. (i), we get

$$v+x\frac{dv}{dx}=\frac{-v^2}{1-v+v^2}$$

Step III Simplify the equation of Step II and convert it into variable separable form and get the required solution after integration.

$$x\frac{dv}{dx}=\frac{-v-v^3}{v^2-v+1}\quad\Rightarrow\quad\frac{v^2-v+1}{v(v^2+1)}dv=-\frac{dx}{x}$$

$$\Rightarrow \quad \left(\frac{1}{v}-\frac{1}{v^2+1}\right)dv=-\frac{dx}{x}$$

On integrating, we get $\log v-\tan^{-1}v=-\log x+C$

$$\Rightarrow \log\left(\frac{y}{x}\right)-\tan^{-1}\left(\frac{y}{x}\right)=-\log x+C\Rightarrow \log y=\tan^{-1}\left(\frac{y}{x}\right)+C$$

TRY THESE

1. The solution of the differential equation $\frac{dy}{dx}=\frac{y}{x}+\frac{\phi(y/x)}{\phi'(y/x)}$, is

(a) $\phi(y/x)=kx$ (b) $x\,\phi(y/x)=k$ (c) $\phi(y/x)=ky$ (d) $y\,\phi(y/x)=k$

2. A curve passes through the point $\left(1,\frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x}+\sec\left(\frac{y}{x}\right)$, $x>0$. Then, the equation of the curve is

(a) $\sin\left(\frac{y}{x}\right)=\log x+\frac{1}{2}$ (b) $\operatorname{cosec}\left(\frac{y}{x}\right)=\log x+2$

(c) $\sec\left(\frac{2y}{x}\right)=\log x+2$ (d) $\cos\left(\frac{2y}{x}\right)=\log x+\frac{1}{2}$

3. If $x\frac{dy}{dx}=y(\log y-\log x+1)$, then the solution of the equation is

(a) $\log\left(\frac{x}{y}\right)=Cy$ (b) $\log\left(\frac{y}{x}\right)=Cx$

(c) $x\log\left(\frac{y}{x}\right)=Cy$ (d) $y\log\left(\frac{x}{y}\right)=Cx$

1. (a) **2.** (a) **3.** (b)

BRAiN TEASERS

To boost up your rank in JEE Main and Advanced

1. If $a=\dfrac{\sqrt{5}+1}{\sqrt{5}-1}$ and $b=\dfrac{\sqrt{5}-1}{\sqrt{5}+1}$, then find the value of $\dfrac{a^2+ab+b^2}{a^2-ab+b^2}$.

2. If $\dfrac{1}{\sqrt[3]{9}+\sqrt[3]{3}+1}=A\cdot\sqrt[3]{9}+B\cdot\sqrt[3]{3}+C$, then find the value of $(A+B+C)$.

3. If $a+b+c=0$, then find the value of $\dfrac{1}{a^2+b^2-c^2}+\dfrac{1}{b^2+c^2-a^2}+\dfrac{1}{a^2+c^2-b^2}$.

4. If $x^2+x=5$, then find the value of $(x+3)^3+\dfrac{1}{(x+3)^3}$.

5. If $\dfrac{x-a}{b+c}+\dfrac{x-b}{c+a}+\dfrac{x-c}{a+b}=3$, then find the value of x.

6. If $a^2+b^2+c^2+3=2(a-b-c)$, then find the value of $2a-b+c$.

7. Find the value of $\dfrac{1}{2^2-1}+\dfrac{1}{4^2-1}+\dfrac{1}{6^2-1}+\ldots+\dfrac{1}{50^2-1}$.

8. If $pqr=1$, then find the value of $\dfrac{1}{1+p+q^{-1}}+\dfrac{1}{1+q+r^{-1}}+\dfrac{1}{1+r+p^{-1}}$.

9. If $x+\dfrac{1}{x}=1$, then find the value of $x^{97}+x^{96}+x^{95}+x^{94}+x^{93}+x^{92}$.

10. If $\sqrt{\dfrac{x-a}{x-b}}+\dfrac{a}{x}=\sqrt{\dfrac{x-b}{x-a}}+\dfrac{b}{x}$, then find the value of x.

Solutions

1. We have, $a=\dfrac{\sqrt{5}+1}{\sqrt{5}-1}$ and $b=\dfrac{\sqrt{5}-1}{\sqrt{5}+1}$

$$\therefore\quad a+b=\frac{\sqrt{5}+1}{\sqrt{5}-1}+\frac{\sqrt{5}-1}{\sqrt{5}+1}=\frac{(\sqrt{5}+1)^2+(\sqrt{5}-1)^2}{(\sqrt{5}-1)(\sqrt{5}+1)}$$

$$=\frac{5+1+2\sqrt{5}+5+1-2\sqrt{5}}{5-1}=3$$

Also, $\quad ab=\dfrac{\sqrt{5}+1}{\sqrt{5}-1}\times\dfrac{\sqrt{5}-1}{\sqrt{5}+1}=1$

Now, $\dfrac{a^2+ab+b^2}{a^2-ab+b^2}=\dfrac{(a^2+b^2)+ab}{(a^2+b^2)-ab}$

$$=\frac{(a+b)^2-ab}{(a+b)^2-3ab}=\frac{(3)^2-1}{(3)^2-3(1)}=\frac{9-1}{9-3}=\frac{8}{6}=\frac{4}{3}$$

2. We have, $\dfrac{1}{\sqrt[3]{9}+\sqrt[3]{3}+1}=A\cdot\sqrt[3]{9}+B\cdot\sqrt[3]{3}+C$

$$\Rightarrow\quad\frac{(3^{1/3}-1)}{(3^{1/3}-1)\,[(3^{1/3})^2+3^{1/3}+1]}=A\cdot\sqrt[3]{9}+B\cdot\sqrt[3]{3}+C$$

$$\Rightarrow\quad\frac{3^{1/3}-1}{(3^{1/3}-1)\,[(3^{1/3})^2+3^{1/3}\times1+(1)^2]}=A\cdot\sqrt[3]{9}+B\cdot\sqrt[3]{3}+C$$

$$\Rightarrow\quad\frac{3^{1/3}-1}{(3^{1/3})^3-(1)^3}=A\cdot\sqrt[3]{9}+B\cdot\sqrt[3]{3}+C$$

$$\Rightarrow\quad\frac{3^{1/3}-1}{3-1}=A\cdot\sqrt[3]{9}+B\cdot\sqrt[3]{3}+C$$

$$\Rightarrow\quad\frac{1}{2}\sqrt[3]{3}-\frac{1}{2}=A\cdot\sqrt[3]{9}+B\cdot\sqrt[3]{3}+C$$

On comparing, we get $A=0$, $B=\dfrac{1}{2}$ and $C=-\dfrac{1}{2}$

$$\therefore\quad A+B+C=0+\frac{1}{2}-\frac{1}{2}=0$$

3. We have, $\quad a+b+c=0$

$$\Rightarrow\qquad a+b=-c\Rightarrow(a+b)^2=c^2$$

$\Rightarrow \quad a^2 + b^2 + 2ab = c^2 \quad \Rightarrow \quad a^2 + b^2 - c^2 = -2ab$

Similarly, $b^2 + c^2 - a^2 = -2bc$

and $a^2 + c^2 - b^2 = -2ac$

Now, $\dfrac{1}{a^2+b^2-c^2} + \dfrac{1}{b^2+c^2-a^2} + \dfrac{1}{a^2+c^2-b^2}$

$= \dfrac{1}{-2ab} + \dfrac{1}{-2bc} + \dfrac{1}{-2ac}$

$= \dfrac{-1}{2}\left[\dfrac{1}{ab} + \dfrac{1}{bc} + \dfrac{1}{ca}\right] = \dfrac{-1}{2}\left[\dfrac{a+b+c}{abc}\right]$

$= \dfrac{-1}{2} \times \dfrac{0}{abc} \qquad [\because a + b + c = 0]$

$= 0$

4. Let $x + 3 = p \quad \Rightarrow \quad x = p - 3$

It is given that $x^2 + x = 5$

$\Rightarrow \quad (p-3)^2 + (p-3) = 5$

$\Rightarrow \quad p^2 + 9 - 6p + p - 3 = 5$

$\Rightarrow \quad p^2 - 5p + 1 = 0$

$\Rightarrow \quad p^2 + 1 = 5p$

$\Rightarrow \quad p + \dfrac{1}{p} = 5$

On cubing both sides, we get

$$p^3 + \frac{1}{p^3} + 3p \cdot \frac{1}{p}\left(p + \frac{1}{p}\right) = 125$$

$\Rightarrow \quad p^3 + \dfrac{1}{p^3} + 3 \times 5 = 125$

$\Rightarrow \quad p^3 + \dfrac{1}{p^3} = 110$

$\therefore \quad (x+3)^3 + \dfrac{1}{(x+3)^3} = 110$

5. We have, $\dfrac{x-a}{b+c} + \dfrac{x-b}{c+a} + \dfrac{x-c}{a+b} = 3$

$\Rightarrow \quad \dfrac{x-a}{b+c} - 1 + \dfrac{x-b}{c+a} - 1 + \dfrac{x-c}{a+b} - 1 = 0$

$\Rightarrow \quad \dfrac{x-a-b-c}{b+c} + \dfrac{x-b-c-a}{c+a} + \dfrac{x-c-a-b}{a+b} = 0$

$\Rightarrow \quad (x-a-b-c)\left[\dfrac{1}{b+c} + \dfrac{1}{c+a} + \dfrac{1}{a+b}\right] = 0$

$\Rightarrow \quad x - a - b - c = 0 \Rightarrow x = a + b + c$

6. We have, $a^2 + b^2 + c^2 + 3 = 2(a - b - c)$

$\Rightarrow \quad a^2 - 2a + b^2 + 2b + c^2 + 2c + 3 = 0$

$\Rightarrow \quad (a^2 - 2a + 1) + (b^2 + 2b + 1) + (c^2 + 2c + 1) = 0$

$\Rightarrow \quad (a-1)^2 + (b+1)^2 + (c+1)^2 = 0$

$\therefore \quad a - 1 = 0, b + 1 = 0, c + 1 = 0$

$\Rightarrow \quad a = 1, b = -1, c = -1$

Now, $2a - b + c = 2(1) - (-1) + (-1) = 2$

7. We have, $\dfrac{1}{2^2-1} + \dfrac{1}{4^2-1} + \dfrac{1}{6^2-1} + \ldots + \dfrac{1}{50^2-1}$

$= \dfrac{1}{(2-1)(2+1)} + \dfrac{1}{(4-1)(4+1)} + \dfrac{1}{(6-1)(6+1)} + \ldots + \dfrac{1}{(50-1)(50+1)}$

$= \dfrac{1}{1\times 3} + \dfrac{1}{3\times 5} + \dfrac{1}{5\times 7} + \ldots + \dfrac{1}{49\times 51}$

$= \dfrac{1}{2}\left[\dfrac{2}{1\times 3} + \dfrac{2}{3\times 5} + \dfrac{2}{5\times 7} + \ldots + \dfrac{2}{49\times 51}\right]$

$= \dfrac{1}{2}\left[\dfrac{3-1}{1\times 3} + \dfrac{5-3}{3\times 5} + \dfrac{7-5}{5\times 7} + \ldots + \dfrac{51-49}{49\times 51}\right]$

$= \dfrac{1}{2}\left[\dfrac{1}{1} - \dfrac{1}{3} + \dfrac{1}{3} - \dfrac{1}{5} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots + \dfrac{1}{49} - \dfrac{1}{51}\right]$

$= \dfrac{1}{2}\left[1 - \dfrac{1}{51}\right] = \dfrac{1}{2} \times \dfrac{50}{51} = \dfrac{25}{51}$

8. We have, $\dfrac{1}{1+p+q^{-1}} + \dfrac{1}{1+q+r^{-1}} + \dfrac{1}{1+r+p^{-1}}$

$= \dfrac{1}{1+p+\frac{1}{q}} + \dfrac{1}{1+q+\frac{1}{r}} + \dfrac{1}{1+r+\frac{1}{p}}$

$= \dfrac{q}{q+pq+1} + \dfrac{1}{1+q+pq} + \dfrac{1}{1+\frac{1}{pq}+\frac{1}{p}} \qquad [\because pqr = 1]$

$= \dfrac{q}{pq+q+1} + \dfrac{1}{pq+q+1} + \dfrac{pq}{pq+q+1} = \dfrac{pq+q+1}{pq+q+1}$

$= 1$

9. We have, $x + \dfrac{1}{x} = 1$

$\Rightarrow x^2 + \dfrac{1}{x^2} + 2 \cdot x \cdot \dfrac{1}{x} = 1 \qquad$ [on squaring both sides]

$\Rightarrow \quad x^2 + \dfrac{1}{x^2} + 2 = 1 \Rightarrow x^2 + \dfrac{1}{x^2} = -1$

$\Rightarrow \quad x\left(x^2 + \dfrac{1}{x^2}\right) = -x \quad$ [multiplying both sides by x]

$\Rightarrow \quad x^3 + \dfrac{1}{x} = -x \Rightarrow x^3 + \dfrac{1}{x} + x = 0$

$\Rightarrow \quad x^3 + 1 = 0 \qquad \left[\because x + \dfrac{1}{x} = 1\right]$

$\Rightarrow \quad x^3 = -1$

Now, $x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + x^{92}$

$= x^{97} + x^{94} + x^{96} + x^{93} + x^{95} + x^{92}$

$= x^{94}(x^3 + 1) + x^{93}(x^3 + 1) + x^{92}(x^3 + 1)$

$= x^{94} \times 0 + x^{93} \times 0 + x^{92} \times 0 = 0 + 0 + 0 = 0$

10. Let $x - a = p$ and $x - b = q$. Then,

$\sqrt{\dfrac{x-a}{x-b}} + \dfrac{a}{x} = \sqrt{\dfrac{x-b}{x-a}} + \dfrac{b}{x} \quad \Rightarrow \quad \sqrt{\dfrac{p}{q}} - \sqrt{\dfrac{q}{p}} = \dfrac{b}{x} - \dfrac{a}{x}$

$\Rightarrow \quad \dfrac{p-q}{\sqrt{pq}} = \dfrac{b-a}{x} \quad \Rightarrow \quad \dfrac{(x-a)-(x-b)}{\sqrt{(x-a)(x-b)}} = \dfrac{b-a}{x}$

$\Rightarrow \quad \dfrac{b-a}{\sqrt{(x-a)(x-b)}} = \dfrac{b-a}{x}$

$\Rightarrow \quad \sqrt{(x-a)(x-b)} = x$

$\Rightarrow \quad (x-a)(x-b) = x^2$

$\Rightarrow \quad x^2 - (a+b)x + ab = x^2$

$\Rightarrow \quad (a+b)x = ab$

$\Rightarrow \quad x = \dfrac{ab}{a+b}$

CONCEPT MAP your Revision Tool

BINOMIAL THEOREM

❶ BINOMIAL THEOREM

If x and a are real numbers, then for all $n \in N$,

$(x+a)^n = {}^nC_0x^na^0 + {}^nC_1x^{n-1}a^1 + {}^nC_2x^{n-2}a^2 + \ldots + {}^nC_rx^{n-r}a^r + \ldots + {}^nC_{n-1}x^1a^{n-1} + {}^nC_nx^0a^n$

or $\quad (x+a)^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}a^r$

The above expansion is also valid when x and a are complex.

❷ PROPERTIES OF BINOMIAL EXPANSION

Following are some properties of binomial expansion

Ⓐ We have,

$$(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}a^r$$

Since, r can have values from 0 to n, the total number of terms in the expansion is $(n+1)$.

Ⓑ The sum of the indices of x and a in each term is n.

Ⓒ We have,

$${}^nC_r = {}^nC_{n-r},\ r = 0, 1, 2, \ldots, n$$

$$\Rightarrow \quad {}^nC_0 = {}^nC_n,\ {}^nC_1 = {}^nC_{n-1}, \ldots$$

So, the coefficient of terms equidistant from the beginning and the end are equal. These coefficients are known as the binomial coefficients.

Ⓓ Replacing a by $-a$, in the expansion of $(x+a)^n$, we get

$$(x-a)^n = {}^nC_0x^na^0 - {}^nC_1x^{n-1}a^1 + {}^nC_2x^{n-2}a^2 - {}^nC_3x^{n-3}a^3 + \ldots + (-1)^r\ {}^nC_rx^{n-r}a^r + \ldots + (-1)^n\ {}^nC_nx^0a^n$$

i.e. $(x-a)^n = \sum_{r=0}^{n} (-1)^r\ {}^nC_rx^{n-r}a^r$

Thus, the terms in the expansion of $(x-a)^n$ are alternatively positive and negative, the last term is positive or negative according as n is even or odd.

Ⓔ Putting $x=1$ and $a=x$ in the expansion of $(x+a)^n$, we get

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n$$

i.e. $\quad (1+x)^n = \sum_{r=0}^{n} {}^nC_rx^r$

This is the expansion of $(1+x)^n$ in ascending powers of x.

Ⓕ Putting $a=1$ in the expansion of $(x+a)^n$, we get

$$(1+x)^n = {}^nC_0x^n + {}^nC_1x^{n-1} + {}^nC_2x^{n-2} + \ldots + {}^nC_rx^{n-r} + \ldots + {}^nC_{n-1}x + {}^nC_n$$

i.e. $\quad (1+x)^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}$

This is the expansion of $(1+x)^n$ in descending powers of x.

Ⓖ Putting $x=1$ and $a=-x$ in the expansion of $(x+a)^n$, we get

$$(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - {}^nC_3x^3 + \ldots + (-1)^r\ {}^nC_rx^r + \ldots + (-1)^n\ {}^nC_nx^n$$

i.e. $\quad (1-x)^n = \sum_{r=0}^{n} (-1)^r\ {}^nC_rx^r$

Ⓗ The coefficient of $(r+1)$th term in the expansion of $(1+x)^n$ is ${}^nC_r$.

Ⓘ The coefficient of x^r in the expansion of $(1+x)^n$ is ${}^nC_r$.

Ⓙ We have,

$$(x+a)^n + (x-a)^n = 2\{{}^nC_0x^na^0 + {}^nC_2x^{n-2}a^2 + \ldots\}$$

and $(x+a)^n - (x-a)^n = 2\{{}^nC_1x^{n-1}a^1 + {}^nC_3x^{n-3}a^3 + \ldots\}$

Note If n is odd, then $\{(x+a)^n + (x-a)^n\}$ and $\{(x+a)^n - (x-a)^n\}$ both have the same number of terms equal to $\left(\frac{n+1}{2}\right)$ whereas if n is even, then $\{(x+a)^n + (x-a)^n\}$ has $\left(\frac{n}{2}+1\right)$ terms and $\{(x+a)^n - (x-a)^n\}$ has $\left(\frac{n}{2}\right)$ terms.

❸ GENERAL TERM IN A BINOMIAL EXPANSION

We have, $(x+a)^n = {}^nC_0x^na^0 + {}^nC_1x^{n-1}a^1 + \ldots + {}^nC_rx^{n-r}a^r + \ldots + {}^nC_nx^0a^n$

We observe that $(r+1)$th term is given by

$$T_{r+1} = {}^nC_rx^{n-r}a^r$$

This is called the general term.

In the binomial expansion of $(x-a)^n$, the general term is given by

$$T_{r+1} = (-1)^r\ {}^nC_rx^{n-r}a^r$$

In the binomial expansion of $(1+x)^n$, we have

$$T_{r+1} = {}^nC_rx^r$$

In the binomial expansion of $(1-x)^n$, we have

$$T_{r+1} = (-1)^r\ {}^nC_rx^r$$

In the binomial expansion of $(x+a)^n$, the mth term from the end is $\{(n+1)-m+1\}$ i.e. $(n-m+2)$th term from the beginning.

GREATEST TERM

For the binomial expansion of $(x+a)^n$, we have

$T_{r+1} = {}^nC_rx^{n-r}a^r$ and $T_r = {}^nC_{r-1}x^{n-r+1}a^{r-1}$.

$$\therefore \frac{T_{r+1}}{T_r} = \frac{{}^nC_rx^{n-r}a^r}{{}^nC_{r-1}x^{n-r+1}a^{r-1}}$$

$$\Rightarrow \frac{T_{r+1}}{T_r} = \frac{n!}{(n-r)!\,r!} \times \frac{(r-1)!\,(n-r+1)!}{n!} \cdot \frac{a}{x}$$

$$\Rightarrow \frac{T_{r+1}}{T_r} = \frac{n-r+1}{r} \cdot \frac{a}{x}$$

Now, $\quad T_{r+1} >, =, < T_r$

$$\Rightarrow \quad \frac{T_{r+1}}{T_r} >, =, < 1 \Rightarrow \frac{n-r+1}{r} \cdot \frac{a}{x} >, =, < 1$$

$$\Rightarrow \left\{\left(\frac{n+1}{r}\right) - 1\right\}\frac{a}{x} >, =, < 1$$

$$\Rightarrow \quad \frac{n+1}{r} - 1 >, =, < \left|\frac{x}{a}\right|$$

$$\Rightarrow \quad \left(\frac{n+1}{r}\right) >, =, < \left(1 + \left|\frac{x}{a}\right|\right)$$

$$\Rightarrow \quad \frac{n+1}{1+\left|\frac{x}{a}\right|} >, =, < r \quad \Rightarrow \quad r <, =, > \left(\frac{n+1}{1+\left|\frac{x}{a}\right|}\right)$$

Thus, $T_{r+1} >, =, < T_r$ according as

$$r <, =, > \left(\frac{n+1}{1+\left|\frac{x}{a}\right|}\right) \qquad \ldots(i)$$

Now, two cases arise

***Case* I** When $\left(\frac{n+1}{1+\left|\frac{x}{a}\right|}\right)$ is an integer .

Let $\frac{n+1}{1+\left|\frac{x}{a}\right|}=m$. Then, from Eq. (i), we have

$T_{r+1}>T_r$ for $r=1, 2, 3, \ldots, (m-1)$ … (ii)

$T_{r+1}=T_r$ for $m=m$ …(iii)

and $T_{r+1}<T_r$ for $r=m+1, \ldots, n$ …(iv)

$\therefore T_2>T_1, T_3>T_2, T_4>T_3, \ldots, T_m>T_{m-1}$

$T_{m+1}=T_m$ and $T_{m+2}<T_{m+1}, T_{m+3}<T_{m+2}, \ldots T_{n+1}<T_n$

$\Rightarrow T_1<T_2<\ldots<T_{m-1}<T_m=T_{m+1}>T_{m+2}>\ldots T_n$

This shows that mth and $(m+1)$th terms are greatest terms.

***Case* II** When $\left(\frac{n+1}{1+\left|\frac{x}{a}\right|}\right)$ is not an integer.

Let m be the integral part of $\frac{n+1}{1+\left|\frac{x}{a}\right|}$. Then, from Eq. (i), we have

$T_{r+1}>T_r$ for $r=1, 2, \ldots, m$ …(v)

and $T_{r+1}<T_r$ for $r=m+1, m+2, \ldots, n$ …(vi)

$\therefore T_2>T_1, T_3>T_2, \ldots, T_{m+1}>T_m$

and $T_{m+2}<T_{m+1}, T_{m+3}<T_{m+2}, \ldots, T_{n+1}<T_n$

$\Rightarrow T_1<T_2<T_3<\ldots<T_m<T_{m+1}>T_{m+2}>T_{m+3}\ldots T_n$

$\Rightarrow (m+1)$th term is the greatest term.

GREATEST COEFFICIENT

In the binomial expansion of $(x+a)^n$, the coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are known as the binomial coefficients. If n is even, then ${}^nC_{n/2}$ is the greatest.

❹ MIDDLE TERM IN A BINOMIAL EXPANSION

If n is an even natural number, then in the binomial expansion of $(x+a)^n$, $\left(\frac{n}{2}+1\right)$th term is the middle term.

If n is odd natural number, then $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms are the middle terms in the binomial expansion of $(x+a)^n$.

❺ INTEGRAL AND FRACTIONAL PARTS

Following algorithm is used to find the integral part and fractional part of an irrational number of the form $(a+b\sqrt{c})^n$.

ALGORITHM

***Step* I** Write the given expression equal to $I+F$, where I is its integral part and F is the fractional part.

***Step* II** Define G by replacing '+' sign in the given expression by '–'. Note that G always lies between 0 and 1.

***Step* III** Either add G to the expression in Step I or subtract G from the expression in Step I so that RHS is an integer.

***Step* IV** If G is added to the expression in Step I, then $G+F$ will always come out to be equal to 1 i.e. $G=1-F$. If G is subtracted from the expression in Step I, then G will always come out to be equal to F.

***Step* V** Obtain the value of the desired expression after getting F in terms of G.

❻ MULTINOMIAL THEOREM

Using binomial theorem, we have

$$(x+a)^n=\sum_{r=0}^{n}{}^nC_r x^{n-r}a^r, n\in N$$

$$\Rightarrow (x+a)^n=\sum_{r=0}^{n}\frac{n!}{(n-r)!\,r!}x^{n-r}a^r$$

$$\Rightarrow (x+a)^n=\sum_{r+s=n}\frac{n!}{r!\,s!}x^s a^r, \text{ where } s=n-r$$

This result can be generalised in the following form

$$(x_1+x_2+\ldots+x_k)^n=\sum_{r_1+r_2+\ldots+r_k=n}\frac{n!}{r_1!r_2!\ldots r_k!}x_1^{r_1}x_2^{r_2}\ldots x_k^{r_k}$$

The general term in the above expansion is

$$\frac{n!}{r_1!r_2!\ldots r_k!}x_1^{r_1}x_2^{r_2}\ldots x_k^{r_k}$$

The number of terms in the above expansion is equal to the number of non-negative integral solution of the equation $r_1+r_2+\ldots+r_k=n$.

The number of such solutions is ${}^{n+k-1}C_{k-1}$.

PARTICULAR CASES

(i) $(x+y+z)^n=\sum_{r+s+t=n}\frac{n!}{r!\,s!\,t!}x^r y^s z^t$

The above expansion has ${}^{n+3-1}C_{3-1}={}^{n+2}C_2$ terms.

(ii) $(x+y+z+u)^n=\sum_{p+q+r+s=n}\frac{n!}{p!\,q!\,r!\,s!}x^p y^q z^r u^s$

There are ${}^{n+4-1}C_{4-1}={}^{n+3}C_3$ terms in the above expansion.

Note The greatest coefficient in the expansion of $(x_1+x_2+\ldots+x_m)^n$ is $\frac{n!}{(q!)^{m-r}[(q+1)!]^r}$, where q and r are the quotient and remainder, respectively when n is divided by m.

❼ PROPERTIES OF BINOMIAL COEFFICIENTS

We have,

$(1+x)^n={}^nC_0+{}^nC_1x+{}^nC_2x^2+\ldots+{}^nC_nx^n$

Let us denote the binomial coefficients

${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_{n-1}, {}^nC_n$

by $C_0, C_1, C_2, \ldots C_{n-1}, C_n$, respectively.

The binomial coefficients $C_0, C_1, C_2, \ldots, C_n$ possess the following properties.

(A) In the expansion of $(1+x)^n$, the coefficients of terms equidistant from the beginning and the end are equal.

(B) The sum of the binomial coefficients in the expansion of $(1+x)^n$ is 2^n.

i.e. $C_0+C_1+C_2+\ldots+C_n=2^n$

(C) The sum of the coefficients of the odd terms in the expansion of $(1+x)^n$ is equal to the sum of the coefficients of the even terms and each is equal to 2^{n-1}.

i.e. $C_1+C_3+C_5+\ldots=C_0+C_2+C_4+\ldots=2^{n-1}$

(D) If $r, n\in N$ such that $r<n$, then

${}^nC_r=\frac{n}{r}\cdot{}^{n-1}C_{r-1}=\frac{n}{r}\cdot\frac{n-1}{r-1}\cdot{}^{n-2}C_{r-2}$ and so on.

(E) $C_0-C_1+C_2-C_3+C_4-\ldots+(-1)^n C_n=0$

❽ BINOMIAL THEOREM FOR ANY INDEX

Let n be a rational number and x be a real number such that $|x|<1$, then

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots+\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r+\ldots\infty$$

❾ SOME IMPORTANT EXPANSIONS

Following are some important expansions

(A) $(1+x)^{-1}=1-x+x^2-x^3+x^4+\ldots+(-1)^r x^r+\ldots$

(B) $(1+x)^{-2}=1-2x+3x^2-4x^3+5x^4+\ldots+(-1)^r(r+1)x^r+\ldots$

(C) $(1+x)^{-3}=1-3x+6x^2-10x^3+\ldots+(-1)^r\frac{(r+1)(r+2)}{2}x^r+\ldots$

(D) $(1-x)^{-1}=1+x+x^2+x^3+\ldots+x^r+\ldots$

(E) $(1-x)^{-2}=1+2x+3x^2+4x^3+5x^4+\ldots+(r+1)x^r+\ldots$

(F) $(1-x)^{-3}=1+3x+6x^2+10x^3+\ldots+\frac{(r+1)(r+2)}{2}x^r+\ldots$

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Rapid CONCEPT REVISION

TARGET 2017

HIGHER ORDER DIFFERENTIATION

Let $y = f(x)$, then $\frac{dy}{dx}$, the derivative of y with respect to x, itself, a function of x and can be differentiated again. In general $\frac{dy}{dx}$ is known as first order derivative of y w.r.t. x and the derivative of $\frac{dy}{dx}$ w.r.t. x termed as the second order derivative of y w.r.t. x and it is will be denoted by $\frac{d^2y}{dx^2}$. Similarly, the derivative of $\frac{d^2y}{dx^2}$ w.r.t. x will be termed as the third order derivative of y w.r.t. x and will be denoted by $\frac{d^3y}{dx^3}$ and so on. The nth order derivative of y w.r.t. x will be denoted by $\frac{d^ny}{dx^n}$. If $y = f(x)$, then the other alternative notations for $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}$ are $y_1, y_2, y_3, \ldots, y_n$

or $y', y'', y''', \ldots, y^{(n)}$

or $D_y, D_y^2, D_y^3, \ldots, D_y^n$ or $f'(x), f''(x), f'''(x), \ldots, f^n(x)$

Second Order Derivative of Parametric Functions

If $x = \phi(t)$ and $y = \Psi(t)$, then

$\frac{dy}{dx} = \frac{\Psi'(t)}{\phi'(t)}$, where $\Psi'(t) = \frac{d}{dt}\{\Psi(t)\}$ and $\phi'(t) = \frac{d}{dt}\{\phi(t)\}$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left\{\frac{\Psi'(t)}{\phi'(t)}\right\} = \frac{d}{dt}\left\{\frac{\Psi'(t)}{\phi'(t)}\right\}\cdot\frac{dt}{dx}$

$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dt}\left\{\frac{\Psi'(t)}{\phi'(t)}\right\}\cdot\frac{dt}{dx}$

Leibnitz Theorem for nth Derivatives of Product of Two Functions

Let $u(x)$ and $v(x)$ be functions possessing derivatives upto nth order. Then, $(uv)_n = u_n(x)v(x) + {}^nC_1u_{n-1}(x)v_1(x) + \ldots + {}^nC_ku_{n-k}(x)v_k(x) + \ldots + {}^nC_nu(x)v_n(x)$

where, $u_k(x) = \frac{d^k}{dx^k}u(x)$ [kth derivative of the function $u(x)$], $1 \le k \le n$ and $v_k(x) = \frac{d^k}{dx^k}v(x), 1 \le k \le n$

nth derivatives of some elementary functions

$\frac{d^n}{dx^n}(x^m) = \frac{m!\,x^{m-n}}{(m-n)!}$, if $n \le m$ and $\frac{d^n}{dx^n}(x^m) = 0$, if $n > m$

$\frac{d^n}{dx^n}(\sin x) = \sin\left(x + \frac{n\pi}{2}\right), \frac{d^n}{dx^n}(\cos x) = \cos\left(x + \frac{n\pi}{2}\right)$

$\frac{d^n}{dx^n}(e^{mx}) = m^ne^{mx}$

REMEMBER

(i) $\frac{d^n}{dx^n}[\sin(ax+b)] = a^n\sin\left(\frac{n\pi}{2} + ax + b\right)$

(ii) $\frac{d^n}{dx^n}[\cos(ax+b)] = a^n\cos\left(\frac{n\pi}{2} + ax + b\right)$

(iii) $\frac{d^n}{dx^n}(ax+b)^m = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$

(iv) $\frac{d^n}{dx^n}[\log(ax+b)] = \frac{(-1)^{n-1}(n-1)a^n}{(ax+b)^n}$

(v) (a) $\frac{d^n}{dx^n}[e^{ax}\sin(bx+c)] = r^ne^{ax}\sin(bx+c+n\phi)$

(b) $\frac{d^n}{dx^n}[e^{ax}\cos(bx+c)] = r^ne^{ax}\cos(bx+c+n\phi)$

where, $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

MASTER STROKES

1. If $y = x + e^x$, then $\dfrac{d^2x}{dy^2}$ is equal to

(a) $\dfrac{1}{(1+e^x)^2}$ (b) $-\dfrac{e^x}{(1+e^x)^2}$ (c) $-\dfrac{e^x}{(1+e^x)^3}$ (d) e^x

2. The second order derivative of $a\sin^3 t$ with respect to $a\cos^3 t$ at $t = \pi/4$, is

(a) $\dfrac{4\sqrt{2}}{3a}$ (b) 2 (c) $\dfrac{1}{12a}$ (d) 0

3. Let $f(x)$ and $g(x)$ be two functions having finite non-zero third order derivatives $f'''(x)$ and $g'''(x)$ for all $x \in R$. If $f(x)\, g(x) = 1$ for all $x \in R$, then $\dfrac{f'''(x)}{f'(x)} - \dfrac{g'''(x)}{g'(x)}$ is equal to

(a) $3\left(\dfrac{f''(x)}{g(x)} - \dfrac{g''(x)}{f(x)}\right)$ (b) $3\left(\dfrac{f''(x)}{f(x)} - \dfrac{g''(x)}{g(x)}\right)$

(c) $3\left(\dfrac{g''(x)}{g(x)} - \dfrac{f''(x)}{g(x)}\right)$ (d) $3\left(\dfrac{f''(x)}{f(x)} - \dfrac{g''(x)}{f(x)}\right)$

4. If $x = \phi(t)$, $y = \Psi(t)$, then $\dfrac{d^2y}{dx^2}$ is equal to

(a) $\dfrac{\phi'(t)\,\psi''(t) - \psi'(t)\phi''(t)}{(\phi'(t))^2}$ (b) $\dfrac{\phi'(t)\,\psi''(t) - \psi'(t)\,\phi''(t)}{(\phi'(t))^3}$

(c) $\dfrac{\phi''(t)}{\psi''(t)}$ (d) $\dfrac{\psi''(t)}{\phi''(t)}$

5. If $y = \sin x + e^x$, then $\dfrac{d^2x}{dy^2}$ equals

(a) $(-\sin x + e^x)^{-1}$ (b) $\dfrac{\sin x - e^x}{(\cos x + e^x)^2}$

(c) $\dfrac{\sin x - e^x}{(\cos x + e^x)^3}$ (d) $\dfrac{\sin x + e^x}{(\cos x + e^x)^3}$

6. If $y^{1/m} = [x + \sqrt{1+x^2}]$, then $(1+x^2)y_2 + xy_1$ is equal to

(a) m^2y (b) my^2 (c) m^2y^2 (d) 0

7. If $f(x) = x^n$, then the value of

$f(1) + \dfrac{f'(1)}{1!} + \dfrac{f''(1)}{2!} + \dfrac{f'''(1)}{3!} + \ldots + \dfrac{f^n(1)}{n!}$, is

(a) n (b) 2^n (c) 2^{n-1} (d) $\dfrac{n(n+1)}{2}$

8. If $f(x)$ is given by

$f(x) = (\cos x + i\sin x)(\cos 3x + i\sin 3x) \ldots$

$\ldots\{\cos(2n-1)\,x + i\sin(2n-1)x\}$, then $f''(x)$ is equal to

(a) $n^2 f(x)$ (b) $-n^4 f(x)$ (c) $-n^2 f(x)$ (d) $n^4 f(x)$

9. Let $f(x) = \sin x$, $g(x) = x^2$ and $h(x) = \log x$.

If $F(x) = (hogof)(x)$, then $F''(x)$ is equal to

(a) $2\,\text{cosec}^3 x$ (b) $2\cot x^2 - 4x^2\,\text{cosec}^2 x^2$

(c) $2x\,\cot x^2$ (d) $-2\,\text{cosec}^2 x$

10. The functions $u = e^x \sin x$; $v = e^x \cos x$ satisfy the equation

(a) $v\dfrac{du}{dx} - u\dfrac{dv}{dx} = u^2 + v^2$ (b) $\dfrac{d^2u}{dx^2} = 2v$

(c) $\dfrac{d^2v}{dx^2} = -2u$ (d) All of these

11. If $x^2 + y^2 = 1$, then

(a) $yy'' - 2(y')^2 + 1 = 0$ (b) $yy'' + (y')^2 - 1 = 0$

(c) $yy'' + (y')^2 + 1 = 0$ (d) $yy'' + 2(y')^2 + 1 = 0$

12. If $f(x) = \begin{vmatrix} x^n & \sin x & \cos x \\ n! & \sin\dfrac{n\pi}{2} & \cos\dfrac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix}$, then the value of

$\dfrac{d^n}{dx^n}\{f(x)\}$ at $x = 0$ for $n = 2m+1$ is

(a) -1 (b) 0

(c) 1 (d) dependent of a

13. Let f be a bijection satisfying $f'(x) = f(x)$. Then, $(f^{-1})''(x)$ is equal to

(a) $\dfrac{-1}{x^2}$ (b) $\dfrac{-1}{x^3}$ (c) $f(x)$ (d) $f^{-1}(x)$

14. Let $f(x)$ be a polynomial of degree 3 such that $f(3) = 1, f'(3) = -1$, $f''(3) = 0$ and $f'''(3) = 12$. Then, the value of $f'(1)$ is

(a) -13 (b) 12 (c) 23 (d) 32

15. Let $f(x) = x^2 + x\,g'(1) + g''(2)$ and $g(x) = x^2 + xf''(2) + f''(3)$. Then,

(a) $f'(1) = 4 + f'(2)$ (b) $g'(2) = 8 + g'(1)$

(c) $g''(2) + f''(3) = 4$ (d) All of these

16. $\dfrac{d^2x}{dy^2}$ is equal to

(a) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$ (b) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$

(c) $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-2}$ (d) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$

17. Let $g(x) = \log f(x)$, where $f(x)$ is twice differentiable positive function on $(0, \infty)$ such that $f(x+1) = x\,f(x)$. Then, for $N = 1, 2, 3, \ldots$, $g''\left(N + \dfrac{1}{2}\right) - g''\left(\dfrac{1}{2}\right)$ is equal to

(a) $-4\left\{1 + \dfrac{1}{9} + \dfrac{1}{25} + \ldots + \dfrac{1}{(2N-1)^2}\right\}$

(b) $4\left\{1 + \dfrac{1}{9} + \dfrac{1}{25} + \ldots + \dfrac{1}{(2N-1)^2}\right\}$

(c) $-4\left\{1 + \dfrac{1}{9} + \dfrac{1}{25} + \ldots + \dfrac{1}{(2N+1)^2}\right\}$

(d) $4\left\{1 + \dfrac{1}{9} + \dfrac{1}{25} + \ldots + \dfrac{1}{(2N+1)^2}\right\}$

18. If $f : R \to R$ is defined by $f(x) = \dfrac{x^2 - ax + 1}{x^2 + ax + 1}$, $0 < a < 2$, then which of the following is true?
(a) $(2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 0$
(b) $(2 - a)^2 f''(1) - (2 + a)^2 f''(-1) = 0$
(c) $f'(1) f'(-1) = (2 - a)^2$
(d) $f'(1) f'(-1) = -(2 + a)^2$

19. If a curve is represented parametrically by the equations $x = 4t^3 + 3$ and $y = 4 + 3t^4$ and $\left(\dfrac{d^2x}{dy^2}\right) \Big/ \left(\dfrac{dx}{dy}\right)^n$ is constant, then the value of n is
(a) 3 (b) 4 (c) 5 (d) 6

20. If $y^2 = P(x)$ is a polynomial of degree 3, then $2\dfrac{d}{dx}\left(y^3 \dfrac{d^2y}{dx^2}\right)$ is equal to
(a) $P(x) + P''(x)$ (b) $P(x)$
(c) $P(x)\, P'''(x)$ (d) a constant

21. If $y = \int_0^x f(t) \sin\{k(x - t)\}\, dt$, then $\dfrac{d^2y}{dx^2} + k^2 y$ equals
(a) 0 (b) y (c) $k f(x)$ (d) $k^2 f(x)$

22. If $f(x) = \begin{vmatrix} x^3 & x^4 & 3x^2 \\ 1 & -6 & 4 \\ P & P^2 & P^3 \end{vmatrix}$, where P is a constant, then $\dfrac{d^3}{dx^3}\{f(x)\}$ is
(a) proportional to x^2 (b) proportional to x
(c) proportional to x^3 (d) a constant

23. If $f(x) = \left| \dfrac{\tan^{-1}\left(\dfrac{e}{x^2}\right)}{\log(ex^2)} \right| + \tan^{-1}\left(\dfrac{3 + 2\log x}{1 - 6\log x}\right)$, then $\dfrac{d^n y}{dx^n}$ is
(a) $\tan^{-1}\{(\log x)^n\}$ (b) 0
(c) 1/2 (d) None of these

24. If $y = \sin^2\alpha + \cos^2(\alpha + \beta) + 2\sin\alpha \sin\beta \cos(\alpha + \beta)$, then $\dfrac{d^3y}{d\alpha^3}$ is
(a) $\dfrac{\sin^3(\alpha + \beta)}{\cos\alpha}$ (b) $\cos(\alpha + 3\beta)$
(c) 0 (d) None of these

25. Observe the following statements
I. If $f(x) = ax^{41} + bx^{-40}$, then $\dfrac{f''(x)}{f(x)} = 1640\, x^2$
II. $\dfrac{d}{dx}\left\{\tan^{-1}\left(\dfrac{2x}{1 - x^2}\right)\right\} = \dfrac{1}{1 + x^2}$
Which of the following is correct?
(a) I is true, but II is false
(b) Both I and II are true
(c) Neither I nor II is true
(d) I is false, but II is true

26. If $y = \log(1 + \sin x)$, then the value of $y_4 + y_3 y_1 + y_2^2$ is equal to
(a) 0 (b) $\sin x$ (c) $\cos x$ (d) 1

27. If $y = \sin px$ and y_n is the nth derivative of y, then $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$ is
(a) 1 (b) 0
(c) -1 (d) None of these

28. If $f''(x) = -f(x)$, $g(x) = f'(x)$, $F(x) = \left(f\left(\dfrac{x}{2}\right)\right)^2 + \left(g\left(\dfrac{x}{2}\right)\right)^2$ and $F(5) = 5$, then $F(10)$ equals
(a) 5 (b) 10 (c) 0 (d) 15

29. If graph of $y = f(x)$ is symmetrical about the Y-axis and that of $y = g(x)$ is symmetrical about the origin and if $h(x) = f(x) - g(x)$, then $\dfrac{d^3h(x)}{dx^3}$ at $x = 0$ is
(a) Cannot be determined
(b) $f(0) - g(0)$
(c) 0
(d) None of the above

30. If $y = (A + Bx)e^{mx} + (m - 1)^{-2} e^x$, then $\dfrac{d^2y}{dx^2} - 2m\dfrac{dy}{dx} + m^2 y$ is equal to
(a) e^x (b) e^{mx} (c) e^{-mx} (d) $e^{(1-m)x}$

31. If $y = x\log\left(\dfrac{x}{a + bx}\right)$, then $x^3 \dfrac{d^2y}{dx^2}$ is equal to
(a) $\left(x\dfrac{dy}{dx} - y\right)^2$ (b) $\dfrac{a^2x}{(a + bx)^2}$
(c) $\left(\dfrac{dy}{dx} - y\right)^2$ (d) $\left(x\dfrac{dy}{dx} + y\right)^2$

32. If $x = \tan\left(\dfrac{1}{a}\log y\right)$, then $(1 + x^2)\dfrac{d^2y}{dx^2} - (a - 2x)\dfrac{dy}{dx}$ is equal to
(a) 0 (b) 1 (c) 2 (d) $2x$

Answers

1. (c)	2. (a)	3. (b)	4. (b)	5. (c)
6. (a)	7. (b)	8. (b)	9. (d)	10. (d)
11. (c)	12. (b)	13. (a)	14. (c)	15. (d)
16. (d)	17. (a)	18. (a)	19. (c)	20. (c)
21. (c)	22. (b)	23. (b)	24. (c)	25. (c)
26. (a)	27. (b)	28. (a).	29. (a)	30. (a)
31. (d)	32. (a)			

MEAN VALUE THEOREM

In calculus, the mean value theorem roughly states that in a given section of a smooth curve, there is a point at which the derivative (slope) of the curve is equal to the 'average' derivative of the section.

Rolle's Mean Value Theorem (RMVT)

If a function $f(x)$ is

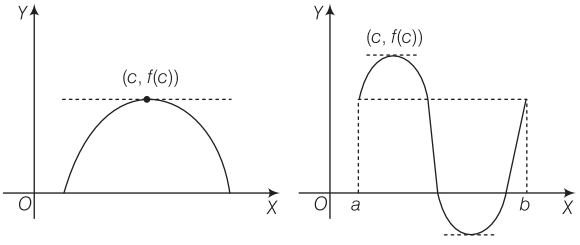
(i) continuous in the closed interval $[a, b]$, i.e. continuous at each point in the interval $[a, b]$,

(ii) differentiable in an open interval (a, b), i.e. differentiable at each point in the open interval (a, b) and

(iii) $f(a) = f(b)$

Then, there exists atleast one point c in the interval (a, b) such that $f'(c) = 0$.

Geometrically, under the assumption of Rolle's theorem, the graph of $f(x)$ starts at point $(a, 0)$ and ends at point $(b, 0)$ as shown in figures.



The conclusion is that there exists atleast one point c between a and b, such that the tangent to the graph at $[c, f(c)]$ is parallel to the X-axis.

REMEMBER

Between any two roots of a polynomial $f(x)$, there is always a root of its derivative $f'(x)$.

Lagrange's Mean Value Theorem (LMVT)

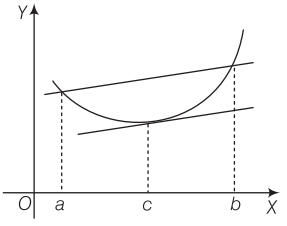
If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$, i.e. continuous at each point in the interval $[a, b]$ and

(ii) differentiable in the open interval (a, b), i.e. differentiable at each point in the interval (a, b).

Then, there exists atleast one point c, where $a < c < b$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Geometrically, any chord of the curve $y = f(x)$, there is a point on the graph, where the tangent is parallel to this chord.



REMEMBER

In the particular case, where $f(a) = f(b)$, the expression $\dfrac{f(b) - f(a)}{b - a}$ becomes zero. Thus, when $f(a) = f(b)$, $f'(c) = 0$ for some c in (a, b). Thus, Rolle's theorem becomes a particular case of the mean value theorem.

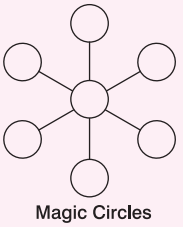
Cauchy's Mean Value Theorem

Cauchy's mean value theorem, also known as the extended mean value theorem, is the more general form of the mean value theorem. It states that if both $f(t)$ and $g(t)$ are continuous functions on the closed interval $[a, b]$, differentiable on the open interval (a, b) and $g'(t)$ is not zero on that open interval, then there exists some c in (a, b), such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Brain Shaker!!

Can you place the numbers 11, 12, 13, 14, 15, 16 and 17, each number in one circle so that any three circles in a line has numbers which total the same sum?



Magic Circles

MASTER STROKES

1. Rolle's theorem is not applicable to the function $f(x) = |x|$ for $-2 \le x \le 2$ because
(a) f is continuous on $[-2, 2]$
(b) f is not derivable at $x = 0$
(c) $f(-2) = f(x)$
(d) f is not a constant function

2. A function f is defined by $f(x) = e^x \sin x$ in $[0, \pi]$. Which of the following is not correct?
(a) f is continuous in $[0, \pi]$
(b) f is differentiable in $(0, \pi)$
(c) $f(0) = f(\pi)$
(d) Rolle's theorem is not applicable to $f(x)$ on $[0, \pi]$

3. If $f(x)$ satisfies the conditions for Rolle's theorem on $[3, 5]$, then $\int_3^5 f(x)\,dx$ equals
(a) 2 (b) -1 (c) 0 (d) $-4/3$

4. If $2a + 3b + 6c = 0$, then the equation $ax^2 + bx + c = 0$ has in the interval $(0, 1)$
(a) atleast one root (b) atmost one root
(c) no root (d) None of these

5. If $0 < a < b < \frac{\pi}{2}$ and $f(a, b) = \frac{\tan b - \tan a}{b - a}$, then
(a) $f(a, b) \ge 2$ (b) $f(a, b) > 1$
(c) $f(a, b) \le 1$ (d) None of these

6. If a, b and c are non-zero real numbers such that $\int_0^1 (1 + \cos^8 x)(ax^2 + bx + c)\,dx = \int_0^2 (1 + \cos^8 x)(ax^2 + bx + c)\,dx = 0$, then the equation $ax^2 + bx + c = 0$ will have
(a) one root between 0 and 1 and other root between 1 and 2
(b) both the roots between 0 and 1
(c) both the roots between 1 and 2
(d) None of the above

7. Between any two real roots of the equation $e^x \sin x - 1 = 0$, the equation $e^x \cos x + 1 = 0$ has
(a) atleast one root (b) atmost one root
(c) exactly one root (d) no root

8. If the functions $f(x)$ and $g(x)$ are continuous on $[a, b]$ and differentiable on (a, b), then in the interval (a, b), the equation
$$\begin{vmatrix} f'(x) & f(a) \\ g'(x) & g(a) \end{vmatrix} = \frac{1}{a - b} \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} \text{ has}$$
(a) atleast one root (b) exactly one root
(c) atmost one root (d) no root

9. If a function is everywhere continuous and differentiable such that $f'(x) \ge 6$ for all $x \in [2, 4]$ and $f(2) = -4$, then
(a) $f(4) < 8$ (b) $f(4) \ge 8$
(c) $f(4) \ge 2$ (d) None of these

10. Let $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex$, where $a, b, c, d, e \in R$ and $f(x) = 0$ has a positive root α. Then,
(a) $f'(x) = 0$ has a root α, such that $0 < \alpha_1 < \alpha_0$
(b) $f''(x) = 0$ has atleast one real root
(c) $f'(x) = 0$ has atleast two real roots
(d) All of the above

11. In $[0, 1]$, Lagrange mean value theorem is not applicable to
(a) $f(x) = \begin{cases} \frac{1}{2} - x, & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2, & x \ge \frac{1}{2} \end{cases}$
(b) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$
(c) $f(x) = x|x|$
(d) $f(x) = |x|$

12. Rolle's theorem hold for the function $f(x) = x^3 + bx^2 + cx$, $1 \le x \le 2$ at the point $4/3$, the values of b and c are
(a) $b = 8, c = -5$ (b) $b = -5, c = 8$
(c) $b = 5, c = -8$ (d) $b = -5, c = -8$

13. Let $f(x)$ satisfy the requirement of Lagrange's mean value theorem in $[0, 2]$. If $f(0) = 0$ and $|f'(x)| \le \frac{1}{2}$ for all $x \in [0, 2]$, then
(a) $f(x) \le 2$
(b) $|f(x)| \le 1$
(c) $f(x) = 2x$
(d) $f(x) = 3$ for atleast one x in $[0, 2]$

14. If $f(x) = x^\alpha \log x$ and $f(0) = 0$, then the value of α for which Rolle's theorem can be applied in $[0, 1]$, is
(a) -2 (b) -1
(c) 0 (d) 1/2

15. If $f(x) = (x - p)(x - q)(x - r)$, where $p < q < r$, are real numbers, then the application of Rolle's theorem on f leads to
(a) $(p + q + r)(pq + qr + rp) = 3$
(b) $(p + q + r)^2 = 3(pq + qr + rp)$
(c) $(p + q + r)^2 > 3(pq + qr + rp)$
(d) $(p + q + r)^2 < 3(pq + qr + rp)$

16. Let $f : [0, 4] \in R$ be a continuous function such that $|f(x)| \le 2$ for all $x \in [0, 4]$ and $\int_0^4 f(t)\,dt = 2$. Then, for all $x \in [0, 4]$, the value of $\int_0^x f(t)\,dt$ lies in the interval
(a) $[-6 + 2x, 10 - 2x]$ (b) $[-12 + 2x, -7 + 2x]$
(c) $[11 - 2x, 17 + 2x]$ (d) $[-8 - 2x, 6 - 2x]$

17. Let f be twice differentiable function satisfying $f(1) = 1$, $f(2) = 4$, $f(3) = 9$, then
(a) $f''(x) = 2$ for all $x \in R$
(b) $f'(x) = 5 = f''(x)$, for some $x \in [1, 3]$
(c) there exists atleast one $x \in (1, 3)$ such that $f''(x) = 2$
(d) None of the above

18. If $f(x)$ is twice differentiable function such that $f(a) = 0$, $f(b) = 2$, $f(c) = -1$, $f(d) = 2$ and $f(e) = 0$, where $a < b < c < d < e$, then the minimum number of zeros of $g(x) = \{f'(x)\}^2 + f''(x)\, f(x)$ in the interval $[a, e]$ is
(a) 7 *(b)* 4 *(c)* 6 *(d)* 3

19. Let $f, g : [-1, 2] \to R$ be continuous functions which are twice differentiable on the interval $(-1, 2)$. Let the values of f and g at the points -1, 0 and 2 be as given in the following table

	$x = -1$	$x = 1$	$x = 2$
$f(x)$	3	6	0
$g(x)$	0	1	-1

In each of the intervals $(-1, 0)$ and $(0, 2)$, the function $(f - 3g)^{\pi}$ never vanishes. Then, the correct statement(s) is/are
(a) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$
(b) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(-1, 0)$
(c) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(0, 2)$
(d) $f'(x) - 3g'(x) = 0$ has exactly two solutions in $(-1, 0)$ and exactly two solutions in $(0, 2)$

20. When the tangent to the curve $y = x \log x$ is parallel to the chord joining the points $(1, 0)$ and (e, e), the value of x is
(a) $\frac{1}{1-e}$ *(b)* $e^{(e-1)(2e-1)}$ *(c)* $e^{\frac{2e-1}{e-1}}$ *(d)* $\frac{e-1}{e}$

21. If the polynomial equation $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$ has two different real roots α and β, then between α and β, the equation $na_n x^{n-1} + (n-1)\, a_{n-1}x^{n-2} + \ldots + a_1 = 0$ has
(a) exactly one root *(b)* atmost one root
(c) atleast one root *(d)* no root

22. The number of real roots of the equation $e^{x-1} + x - 2 = 0$ is/are
(a) 1 *(b)* 2 *(c)* 3 *(d)* 4

23. The distance travelled by a particle upto time x is given by $f(x) = x^3 - 2x + 1$. The time c at which the velocity of the particle is equal to its average velocity between times $x = 1$ s and $x = 2$ s, is
(a) 1.5 s *(b)* $\sqrt{\frac{3}{2}}$ s
(c) $\sqrt{3}$ s *(d)* $\sqrt{\frac{7}{3}}$ s

24. If $f(x)$ is a function given by
$$f(x) = \begin{vmatrix} \sin x & \sin a & \sin b \\ \cos x & \cos a & \cos b \\ \tan x & \tan a & \tan b \end{vmatrix}, \text{ where } 0 < a < b < \frac{\pi}{2}.$$
Then, the equation $f'(x) = 0$ has
(a) atleast one root in (a, b)
(b) atmost one root in (a, b)
(c) exactly one root in (a, b)
(d) no root in (a, b)

25. Suppose α, β and θ are angles satisfying $0 < \alpha < \theta < \beta < \frac{\pi}{2}$. Then, the value of $\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha}$ is
(a) $\cot\theta$ *(b)* $\tan\theta$ *(c)* $-\cot\theta$ *(d)* $-\tan\theta$

26. If $f(x)$ is differentiable in $[a, b]$ such that $f(a) = 2$, $f(b) = 6$, then there exists atleast one c, $a < c < b$, such that $(b^3 - a^3)f'(c)$ is equal to
(a) c^2 *(b)* $2c^2$
(c) $-3c^2$ *(d)* $12c^2$

27. If $g(x) = \frac{x+2}{x-1}$ and the line $3x + y - 10 = 0$, then the line is
(a) tangent to $g(x)$ *(b)* normal to $g(x)$
(c) chord of $g(x)$ *(d)* None of these

28. Consider, the function $f(x) = \begin{cases} x\sin\frac{\pi}{x}, & x > 0 \\ 0, & x = 0 \end{cases}$.
Then, the number of points in $(0, 1)$, where the derivative $f'(x)$ vanishes, is
(a) 0 *(b)* 1
(c) 2 *(d)* infinite

29. If $3(a + 2c) = 4(b + 3d)$, then the equation $ax^3 + bx^2 + cx + d = 0$ will have
(a) no real solution
(b) atleast one real root in $(-1, 0)$
(c) atleast one real root in $(0, 1)$
(d) None of the above

30. A value of c for which the conclusion of mean value theorem holds for the function $f(x) = \log_e x$ on the interval $[1, 3]$, is
(a) $\frac{1}{2}\log_e 3$ *(b)* $\log_3 e$
(c) $\log_e 3$ *(d)* $2\log_3 e$

Answers

1. (b)	**2.** (d)	**3.** (d)	**4.** (a)	**5.** (b)
6. (a)	**7.** (a)	**8.** (a)	**9.** (b)	**10.** (d)
11. (a)	**12.** (b)	**13.** (b)	**14.** (d)	**15.** (c)
16. (a)	**17.** (c)	**18.** (c)	**19.** (d)	**20.** (a)
21. (c)	**22.** (a)	**23.** (c)	**24.** (a)	**25.** (c)
26. (d)	**27.** (a)	**28.** (d).	**29.** (b)	**30.** (d)

APPLICATION OF DERIVATIVES

Approximations

Let $y = f(x)$ be a given function defined by $f : X \to R$, where $X \subset R$. Let for small increment Δx in independent variable x corresponding increment Δy in dependent variable y.

Then, $y = f(x)$ and $y + \Delta y = f(x + \Delta x)$

$\Rightarrow \quad \Delta y = f(x + \Delta y) - f(x)$

$\therefore \quad \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \dfrac{dy}{dx}$

Algorithm

***Step* I** Define a functional relationship between the independent variable x and dependent variable y by observing the given expression.

***Step* II** Choose a value of x nearest to the value for which we have to find y in such a way that either y is given for the chosen x or y can be easily computed for chosen x.

***Step* III** Denote the value of x at which we have to find y by $x + \Delta x$.

***Step* IV** Find Δx and assume that $dx = \Delta x$

***Step* V** Find $\dfrac{dy}{dx}$ from the relation obtained in Step I.

***Step* VI** Find the value of $\dfrac{dy}{dx}$ by putting the value of x chosen in Step II.

***Step* VII** Find dy by using the relation $dy = \dfrac{dy}{dx}\, dx$.

***Step* VIII** Assume that $\Delta y \cong dy$.

***Step* IX** Find the value of y by putting the value of x chosen in Step II in the relation obtained in Step I.

***Step* X** The approximate value of y is $y + \Delta y$.

Errors

Let $y = f(x)$ be a given function of x. If Δx is an error in x, then the corresponding error Δy in y is given by $\Delta y = \dfrac{dy}{dx}\, \Delta x$.

Absolute Error

The errors Δx in x and Δy in y are known as the absolute errors.

Relative Error

If Δx is an error in x, then $\dfrac{\Delta x}{x}$ is called the relative error in x.

Percentage Error

If Δx is error in x, then $\dfrac{\Delta x}{x} \times 100$ is called the percentage error in x.

Derivative as a Rate Measure

Whenever one quantity y varies with another quantity x, satisfying some rule $y = f(x)$, then $\dfrac{dy}{dx}$ represents the rate of change of y with respect to x.

Let $y = f(x)$ be a function of x. Let Δy be the change in y corresponding to a small Δx in x. Then, $\dfrac{\Delta y}{\Delta x}$ represents the change in y due to a unit change in x.

As $\Delta x \to 0$, the limiting value of this average rate of change of y with respect to x in the interval $[x, x + \Delta x]$ becomes the instantaneous rate of change of y w.r.t. x.

Thus, $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$ = Instantaneous rate of change of y w.r.t. x.

$\Rightarrow \quad \dfrac{dy}{dx}$ = Rate of change of y w.r.t. x $\qquad \left[\because \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx}\right]$

In order to solve problems related to rate measure, we use following steps

***Step* I** Assign a variable to each quantity. Draw a diagram, if needed.

***Step* II** Write given values of variable and their rate of changes with respect to 't'.

***Step* III** Find an equation giving the relationship between the variable.

***Step* IV** Differentiate both sides of this equation with respect to 't'.

***Step* V** Replace the variable and their derivatives by the numerical values found in Step II and solve.

Velocity and Acceleration in Rectilinear and Circular Motion

Rectilinear Motion

Students have already studied velocity and acceleration in depth. Here, we recall only definition of velocity as well as acceleration in terms of derivative.

The velocity of a moving particle is defined as the rate of change of its displacement with respect to time and the acceleration is defined as the rate of change of velocity with respect to time. If the displacement 's' of a particle moving in a straight line at a time 't' is given by $s = f(t)$, then velocity at time 't', $(v) = \dfrac{ds}{dt}$

Acceleration at time 't',

$$(a) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = v \cdot \frac{dv}{ds}$$

Note

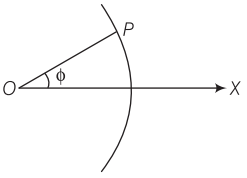
(i) When a particle moving in a straight line comes to rest, then
$$\frac{ds}{dt} = 0 \text{ and } \frac{d^2s}{dt^2} = 0$$
(ii) When a particle moving in a straight line is instantaneously at rest, then
$$\frac{ds}{dt} = 0, \frac{d^2s}{dt^2} \neq 0$$

Circular Motion

Let OX be the initial line and P be the position of a particle moving on a curve at any time 't' such that $\angle XOP = \phi$, then

Angular velocity $= \frac{d\phi}{dt} = \omega$

Angular acceleration $= \frac{d^2\phi}{dt^2} = \frac{d\omega}{dt}$



MASTER STROKES

1. A particle is moving along the curve $x = at^2 + bt + c$. If $ac = b^2$, then the particle would be moving with uniform
(a) rotation (b) velocity
(c) acceleration (d) retardation

2. The distance travelled by a motor car in t seconds after the brakes are applied is s feet, where $s = 22t - 12t^2$. The distance travelled by the car before it stops, is
(a) 10.08 ft (b) 10 ft
(c) 11 ft (d) 11.5 ft

3. A particle moves in a straight line so that $s = \sqrt{t}$, then its acceleration is proportional to
(a) $(\text{velocity})^3$ (b) velocity
(c) $(\text{velocity})^2$ (d) $(\text{velocity})^{3/2}$

4. Side of an equilateral triangle expands at the rate of 2 cm/s. The rate of increasing of its area when each side is 10 cm, is
(a) $10\sqrt{2}$ cm^2/s (b) $10\sqrt{3}$ cm^2/s
(c) 10 cm^2/s (d) 5 cm^2/s

5. A cone whose height is always equal to its diameter, is increasing in volume at the rate of 40 cm^3/s. At what rate is the radius increasing when its circular base area is 1 m^2?
(a) 1 mm/s (b) 0.001 cm/s
(c) 2 mm/s (d) 0.002 cm/s

6. The distance moved by the particle in time t is given by $x = t^3 - 12t^2 + 6t + 8$. At the instant when its acceleration is zero, the velocity is
(a) 42 (b) -42 (c) 48 (d) -48

7. The altitude of a cone is 20 cm and its semi-vertical angle is 30°. If the semi-vertical angle is increasing at the rate of 2° per second, then the radius of the base is increasing at the rate of
(a) 30 cm/s (b) $\frac{160}{3}$ cm/s
(c) 10 cm/s (d) 160 cm/s

8. For what values of x is the rate of increase of $x^3 - 5x^2 + 5x + 8$ is twice the rate of increase of x?
(a) $-3, -\frac{1}{3}$ (b) $-3, \frac{1}{3}$ (c) $3, -\frac{1}{3}$ (d) $3, \frac{1}{3}$

9. The coordinates of the point on the ellipse $16x^2 + 9y^2 = 400$, where the ordinate decreases at the same rate at which the abscissa increases, are
(a) $\left(3, \frac{16}{3}\right)$ (b) $\left(-3, \frac{16}{3}\right)$
(c) $\left(3, -\frac{16}{3}\right)$ (d) $(3, -3)$

10. The radius of the base of a cone is increasing at the rate of 3 cm/min and the altitude is decreasing at the rate of 4 cm/min. The rate of change of lateral surface when the radius is 7 cm and altitude is 24 cm, is
(a) 54 π cm^2/min (b) 7 π cm^2/min
(c) 27 cm^2/min (d) 27 π cm^2/min

11. A particle's velocity v at time t is given by $v = 2e^{2t}\cos\frac{\pi t}{3}$. The least value of t at which the acceleration becomes zero is
(a) 0 (b) $\frac{3}{2}$
(c) $\frac{3}{\pi}\tan^{-1}\left(\frac{6}{\pi}\right)$ (d) $\frac{3}{\pi}\cot^{-1}\left(\frac{6}{\pi}\right)$

12. The surface area of a sphere when its volume is increasing at the same rate as its radius, is
(a) 1 (b) $\frac{1}{2\sqrt{\pi}}$ (c) 4π (d) $\frac{4\pi}{3}$

13. A variable triangle is inscribed in a circle of radius R. If the rate of change of a side is R times the rate of change of the opposite angle, then the angle is
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

14. Two measurements of a cylinder are varying in such a way that the volume is kept constant. If the rate of change of the radius (r) and height (h) are equal in magnitude but opposite in sign, then
(a) $r = 2h$ (b) $h = 2r$ (c) $h = r$ (d) $h = 4r$

15. The weight W of a certain stock of fish is given by $W = nw$, where n is the size of stock and w is the average weight of a fish. If n and w change with time t as $n = 2t^2 + 3$ and $w = t^2 - t + 2$, then the rate of change of W with respect to t at $t = 1$ is
(a) 1 (b) 13 (c) 5 (d) 8

16. A particle moves on the parabola $y^2 = 4\,a\,x$ in such a way that its projection on the Y-axis has a constant velocity. Then, its projection on X-axis moves with
(a) constant velocity (b) constant acceleration
(c) variable velocity (d) variable acceleration

17. The side of a square is equal to the diameter of a circle. If the side and radius change at the same rate, then the ratio of the change of their areas is
(a) 1: π (b) π : 1 (c) 2 : π (d) 1 : 2

18. The points on the curve $12y - x^3 = 0$ whose ordinate and abscissa change at the same rate, are
(a) $\left(-2, -\frac{2}{3}\right), \left(2, \frac{2}{3}\right)$ (b) $\left(-\frac{2}{3}, -2\right), \left(\frac{2}{3}, 2\right)$
(c) $\left(-2, -\frac{2}{3}\right)$ (d) $\left(\frac{2}{3}, 2\right)$

19. A variable ΔABC is inscribed in a circle of diameter x units. If at a particular instant, the rate of change of side 'a' is $\frac{x}{2}$ times the rate of change of the opposite angle A, then A is equal to
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

20. A circular metal plate expands under heating so that its radius increases by 2%. The approximate increase in the area of the plate, if the radius of the plate before heating is 10 cm, is
(a) 2π (b) 3π
(c) π (d) 4π

21. If a ΔABC, inscribed in a fixed circle, is slightly varied in such a way as to have its vertices always on the circle, then $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C}$ is equal to
(a) 0 (b) 1
(c) -1 (d) None of these

22. The percentage error in calculating the volume of a cubical box, if an error of 1% is made in measuring the length of edges of the cube, is
(a) 1% (b) 2%
(c) 3% (d) 1.5%

23. If the percentage error in measuring the surface area of a sphere is α%, then the error in its volume is
(a) $\frac{3}{2}\alpha$% (b) $\frac{2}{3}\alpha$% (c) 3α% (d) 2α%

24. The approximate value of $(1.0002)^{3000}$ is
(a) 1.2 (b) 1.4 (c) 1.6 (d) 1.8

25. The focal length of a mirror is given by $\frac{1}{v} - \frac{1}{u} = \frac{2}{f}$. The equal errors of '$\alpha$' are made in measuring u and v, then the relative error in f is
(a) $\frac{2}{\alpha}$ (b) $\alpha\left(\frac{1}{u} + \frac{1}{v}\right)$ (c) $\alpha\left(\frac{1}{u} - \frac{1}{v}\right)$ (d) 0

26. In a ΔABC, the sides b and c are given. If there is an error ΔA in measuring angle A, then the error Δa in side a is given by
(a) $\frac{S}{2a}\,\Delta A$ (b) $\frac{2S}{a}\,\Delta A$
(c) $bc \sin A\,\Delta A$ (d) None of these

27. If an error of 1° is made in measuring the angle of a sector of radius 30 cm, then the approximate error in its area is
(a) 450 cm^2 (b) 25 $\pi\,\text{cm}^2$ (c) 2.5 $\pi\,\text{cm}^2$ (d) 45 cm^2

28. The semi-vertical angle of a cone is 45°. If the height of the cone is 20.025 units, then its approximate lateral surface area is
(a) $401\sqrt{2}\ \pi$ units (b) $400\sqrt{2}\ \pi$ units
(c) $399\sqrt{2}\ \pi$ units (d) $398\sqrt{2}\ \pi$ units

29. The area of a triangle is computed using the formula $s = \frac{1}{2}bc \sin A$. If the relative errors made in measuring b, c and s are respectively 0.02, 0.01 and 0.13. Then, the approximate error in A when $A = \frac{\pi}{6}$ is
(a) 0.05 rad (b) 0.01 rad
(c) 0.05 degree (d) 0.01 degree

30. If 1° = 0.017 rad, then the approximate value of sin 46° is
(a) 0.7194 (b) $\frac{0.017}{\sqrt{2}}$
(c) $\frac{1.017}{2}$ (d) $\frac{0.7194}{2}$

Answers

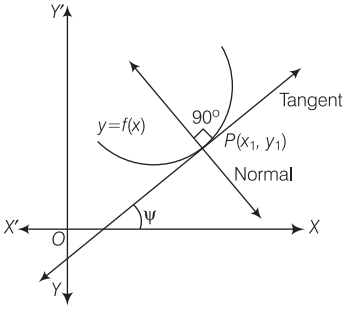
1. (c)	**2.** (a)	**3.** (a)	**4.** (b)	**5.** (d)
6. (b)	**7.** (b)	**8.** (d)	**9.** (a)	**10.** (a)
11. (c)	**12.** (a)	**13.** (c)	**14.** (a)	**15.** (b)
16. (b)	**17.** (c)	**18.** (a)	**19.** (b)	**20.** (d)
21. (a)	**22.** (c)	**23.** (a)	**24.** (c)	**25.** (b)
26. (b)	**27.** (c)	**28.** (a)	**29.** (a)	**30.** (c)

TANGENT AND NORMAL

A line which touches a given curve at a point is known as tangent and a line which is perpendicular to the tangent at the point of contact is known as normal.

Slopes of Tangent and Normal

Let $y = f(x)$ be a continuous curve and $P(x_1, y_1)$ be a point on it. Then, $\left(\frac{dy}{dx}\right)_P$ is the slope of the tangent to the curve $y = f(x)$ at point P, i.e. $\left(\frac{dy}{dx}\right)_P = \tan\psi$ = slope of the tangent at P, where ψ is the angle which the tangent at $P(x_1, y_1)$ makes with the positive direction of X-axis.



The normal to a curve at $P(x_1, y_1)$ is a line perpendicular to the tangent at P and passing through P.

$$\therefore \text{ Slope of the normal at } P = -\frac{1}{\text{Slope of the tangent at } P}$$

$$\text{or slope of the normal at } P = \frac{-1}{\left(\frac{dy}{dx}\right)_P} = -\left(\frac{dx}{dy}\right)_P$$

Note (i) If the tangent at P is parallel to X-axis, then $\psi = 0 \Rightarrow \tan\psi = 0$
$\Rightarrow$ Slope $= 0 \Rightarrow \left(\frac{dy}{dx}\right)_P = 0$

(ii) If the tangent at P is perpendicular to X-axis or parallel to Y-axis, then

$$\psi = \frac{\pi}{2} \Rightarrow \tan\psi = \frac{1}{0} \Rightarrow \left(\frac{dy}{dx}\right)_P = 0 \Rightarrow \left(\frac{dx}{dy}\right)_P = 0$$

Equation of Tangent and Normal

Cartesian Form

We know that the equation of a line passing through a point (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$.

The equation of the tangent at $P(x_1, y_1)$ to the curve $y = f(x)$ is given by

$$y - y_1 = \left(\frac{dy}{dx}\right)_P (x - x_1)$$

The equation of the normal at $P(x_1, y_1)$ to the curve $y = f(x)$ is given by

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_P}(x - x_1)$$

POINTS TO REMEMBER

(i) If $\left(\frac{dy}{dx}\right)_P = \infty$, then the tangent at (x_1, y_1) is parallel to Y-axis and its equation is $x = x_1$.

(ii) If $\left(\frac{dy}{dx}\right)_0 = 0$, then the normal at (x_1, y_1) is parallel to Y-axis and its equation is $x = x_1$.

Parametric Form

If $x = f(t)$ and $y = g(t)$ and $\left|\frac{dy}{dx}\right|_P = \frac{g'(t_0)}{f'(t_0)}$, here coordinates of point P are $x_0 = f(t_0)$, $y_0 = g(t_0)$.

Now, the equation of tangent is

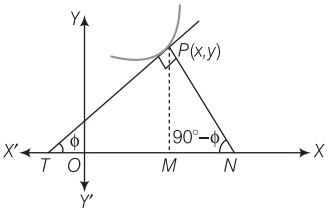
$$(y - g(t_0)) = \frac{g'(t_0)}{f'(t_0)}(x - f(t_0))$$

and equation of normal is

$$(y - g(t_0)) = -\frac{f'(t_0)}{g'(t_0)}(x - f(t_0))$$

Length of Tangent, Normal, Subtangent and Subnormal

Let P be any point on the curve $y = f(x)$.



(i) **Length of Tangent**

$$PT = |y \operatorname{cosec}\phi| = |y\sqrt{1 + \cot^2\phi}| = \left|y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\right|$$

(ii) **Length of Normal**

$$PN = |y \operatorname{cosec}(90° - \phi)|$$
$$= |y \sec\phi| = |y\sqrt{1 + \tan^2\phi}|$$
$$= \left|y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right|$$

(iii) **Length of Subtangent**

$$TM = |y \cot \phi| = \left|\frac{y}{\tan \phi}\right| = \left|y\left(\frac{dx}{dy}\right)\right|$$

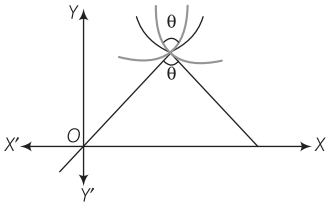
(iv) **Length of Subnormal**

$$MN = |y \cot (90° - \phi)| = |y \tan \phi| = \left|y \cdot \left(\frac{dy}{dx}\right)\right|$$

Angle of Intersection of Two Curves

The angle between two curves is basically angle between the tangents to the curves at the point of intersection of two curves. If m_1 and m_2 are slopes of tangent at the point of contact, then

$$\tan \theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$



Note

(i) If $\theta = \frac{\pi}{2}$, then $m_1 m_2 + 1 = 0 \Rightarrow m_1 \cdot m_2 = -1$

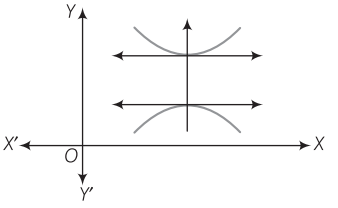
In this situation, the curves are known as **orthogonal curves**.

(ii) If $\theta = 0$, then $m_1 = m_2$, then the curves are **tangential**.

(iii) Length of perpendicular from origin to the tangent is

$$\left|\frac{y_1 - x_1\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2_{(x_1, y_1)}}}\right|$$

(iv) The shortest distance between two curves lies along the common normal i.e. the point where tangents are parallel to each other.



MASTER STROKES

1. If the tangent to the curve $xy + ax + by = 0$ at $(1, 1)$ is inclined at an angle $\tan^{-1} 2$ with X-axis, then

(a) $a = 1, b = 2$ (b) $a = 1, b = -2$
(c) $a = -1, b = 2$ (d) $a = -1, b = -2$

2. If the normal to the curve $y = f(x)$ at the point $(3, 4)$ makes an angle $\frac{3\pi}{4}$ with the positive X-axis, then $f'(3)$ equals

(a) -1 (b) $-\frac{3}{4}$ (c) $-\frac{4}{3}$ (d) 1

3. The tangent to the curve $y = x^3$ at the point $P(t, t^3)$ cuts the curve again at point Q. Then, the coordinates of Q are

(a) $(0, 0)$ (b) $(2t, 4t^3)$
(c) $(2t_1, 8t^3)$ (d) $(-2t, -8t^3)$

4. If the tangent to the curve $2y^3 = ax^2 + x^3$ at the point (a, a) cuts-off intercepts α and β on the coordinate axes such that $\alpha^2 + \beta^2 = 61$, then a equals

(a) ± 30 (b) ± 5
(c) ± 6 (d) ± 61

5. The area of a triangle formed by a tangent to the curve $2xy = a^2$ and the coordinate axes, is

(a) $2a^2$ (b) a^2 (c) $3a^2$ (d) $4a^2$

6. The equation of the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at $x = 0$, is

(a) $x + y = 2$ (b) $x + y = 1$
(c) $x - y = 1$ (d) None of these

7. The angle of intersection of the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $x^2 + y^2 = ab$, is

(a) $\tan^{-1}\left(\frac{a-b}{\sqrt{ab}}\right)$ (b) $\tan^{-1}\left(\frac{a+b}{\sqrt{ab}}\right)$
(c) $\tan^{-1}\left(\frac{a-b}{2\sqrt{ab}}\right)$ (d) $\tan^{-1}\left(\frac{a+b}{2\sqrt{ab}}\right)$

8. If the curves $ax^2 + by^2 = 1$ and $Ax^2 + By^2 = 1$ intersect orthogonally, then

(a) $\frac{1}{a} + \frac{1}{A} = \frac{1}{b} + \frac{1}{B}$ (b) $\frac{1}{a} - \frac{1}{A} = \frac{1}{b} - \frac{1}{B}$
(c) $\frac{1}{a} + \frac{1}{b} = \frac{1}{B} - \frac{1}{A}$ (d) $\frac{1}{a} + \frac{1}{b} = \frac{1}{A} + \frac{1}{B}$

9. The subtangent at any point on the curve $x^m y^n = a^{m+n}$ varies as

(a) $(\text{abscissa})^2$ (b) $(\text{ordinate})^2$
(c) abscissa (d) ordinate

10. Let P be any point on the curve $x^{2/3} + y^{2/3} = a^{2/3}$. Then, the length of the segment of the tangent between the coordinate axes is of length

(a) $3a$ (b) $4a$ (c) $5a$ (d) a

11. The equation of the tangent to the curve $y = (2x - 1)e^{2(1-x)}$ at the point of its maximum is
(a) $y - 1 = 0$ (b) $x = 1$
(c) $x + y - 1 = 0$ (d) $x - y + 1 = 0$

12. If the tangent at any point on the curve $x^4 + y^4 = c^4$ cuts-off intercepts a and b on the coordinate axes, then the value of $a^{-4/3} + b^{-4/3}$ is
(a) $c^{-4/3}$ (b) $c^{-1/2}$ (c) $c^{1/2}$ (d) $c^{4/3}$

13. If the curve with equation of the form $y = ax^4 + bx^3 + c + cx + d$ has zero gradient at the point $(0, 1)$ and also touches the X-axis at the point $(-1, 0)$, then the value of x for which the curve has a negative gradient, are
(a) $x > -1$ (b) $x < 1$
(c) $x < -1$ (d) $-1 \le x \le 1$

14. For the curve $y = ce^{x/a}$, which of the following is incorrect?
(a) Subtangent is constant
(b) Subnormal varies as the square of the ordinate
(c) Tangent at (x_1, y_1) on the curve intersects the X-axis at a distance of $(x_1 - a)$ from the origin
(d) Equation of normal at the point where the curve cuts Y-axis is $cy + ax = c$

15. If m is the slope of the tangent to the curve $e^y = 1 + x^2$, then
(a) $|m| > 1$ (b) $m < 1$
(c) $|m| < 1$ (d) $|m| \le 1$

16. The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the point $(c - 1, e^{c-1})$ and $(c + 1, e^{c+1})$
(a) on the left of $x = c$ (b) on the right of $x = c$
(c) at no point (d) at all points

17. If $x + y = k$ is normal to $y^2 = 12x$, then k is
(a) 3 (b) 9
(c) -3 (d) -9

18. The lengths of tangent, subtangent, normal and subnormal for the curve $y = x^2 + x - 1$ at $(1, 1)$ are A, B, C and D respectively, then their increasing order is
(a) B, D, A, C (b) B, A, C, D
(c) A, B, C, D (d) B, A, D, C

19. If $y = f(x)$ is the equation of the line touching the line $y = 2x + 3$ at $x = 2$, then
(a) $f'(2) = 3$ (b) $2f(2) = 7f'(2)$
(c) $f(2) + f'(2) + f''(2) = 2$ (d) None of these

20. The slope of the tangent to the curve $y = \int_0^x \frac{1}{1 + t^3}\,dt$ at the point where $x = 1$, is
(a) $\frac{1}{2}$ (b) 1
(c) $\frac{1}{4}$ (d) non-existent

21. If the curve $y = x^2 + bx + c$ touches the line $y = x$ at the point $(1, 1)$, then the set of values of x for which the curve has a negative gradient, is
(a) $\left(-\infty, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, \infty\right)$
(c) $\left(-\infty, -\frac{1}{2}\right)$ (d) $\left(-\frac{1}{2}, \infty\right)$

22. If the parabolas $y = x^2 + ax + b$ and $y = x(c - x)$ touch each other at the point $(1, 0)$, then $a + b + c$ equals
(a) 1 (b) -1
(c) 0 (d) None of these

23. The curve $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 2$ touches the line $\frac{x}{a} + \frac{y}{b} = 2$ at the point
(a) (b, a) (b) (a, b) (c) $(1, 1)$ (d) $\left(\frac{1}{b}, \frac{1}{a}\right)$

24. The abscissae of the point on the curve $xy = (a + x)^2$, the normal at which cuts-off numerically equal intercepts from the axes of coordinates, is
(a) $\frac{a}{\sqrt{2}}$ (b) a (c) $\sqrt{2}a$ (d) $-\sqrt{2}a$

25. Let $f(x) = \sin x - \tan x$, $x \in \left(0, \frac{\pi}{2}\right]$, then tangent drawn to the curve $y = f(x)$ at any point will
(a) lie above the curve
(b) lie below the curve
(c) nothing can be said
(d) be parallel to a fixed line

26. If the tangent to the curve $xy + ax + by = 0$ at $(1, 1)$ makes an angle $\tan^{-1} 2$ with X-axis, then $\frac{a + b}{ab}$ is equal to
(a) 0 (b) $\frac{1}{2}$
(c) $-\frac{1}{2}$ (d) None of these

27. The slope of the tangent to the curve $y = \int_x^{x^2} \cos^{-1} t^2\,dt$ at $x = 2^{-1/4}$, is
(a) $\left(\frac{\sqrt[4]{8}}{2} - \frac{3}{4}\right)\pi$ (b) $\left(\frac{\sqrt[4]{8}}{3} - \frac{1}{4}\right)\pi$
(c) $\left(\frac{\sqrt[5]{8}}{4} - \frac{1}{3}\right)\pi$ (d) None of these

28. The equation of a curve is $y = f(x)$. The tangents at $(1, f(1))$, $(2, f(2))$ and $(3, f(3))$ makes angles $\frac{\pi}{6}, \frac{\pi}{3}$ and $\frac{\pi}{4}$ respectively with the positive direction of the X-axis. Then, the value of $\int_2^3 f'(x)\,f''(x)dx + \int_1^3 f''(x)\,dx$ is equal to

(a) $-\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$
(c) 0 (d) None of these

29. Let C be curve $y^3 - 3xy + 2 = 0$. If H is the set of points on the curve C, where the tangent is parallel to X-axis and V is the set of points on C, where the tangent is parallel to Y-axis, then
(a) $H = \{(x, y) : y = 0, x \in R\}, V = \{(1, 1)\}$
(b) $H = \{(x, y) : x = 0, y \in R\}, V = \{(1, 1)\}$
(c) $H = \phi, V = \{(1, 1)\}$
(d) $H = \{(1, 1)\}, V = \{(x, y) : y = 0, x \in R\}$

30. If $\sin^{-1}\theta$ is the acute angle between the curves $x^2 + y^2 = 4x$ and $x^2 + y^2 = 8$ at (2, 2), then $\sin\theta$ equals
(a) 1 (b) 0 (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{\sqrt{3}}{2}$

31. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the X-axis, is
(a) $y = 2$ (b) $y = 3$ (c) $y = 0$ (d) $y = 1$

32. The intercept on X-axis made by tangents to the curve $y = \int_0^x |t|\, dt$, $x \in R$, which are parallel to the line $y = 2x$, are equal to
(a) ± 1 (b) ± 2 (c) ± 3 (d) ± 4

33. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point (1, 3) is
(a) 4 (b) 6 (c) 8 (d) 2

34. The normal to the curve $x^2 + 2xy - 3y^2 = 0$, at (1, 1)
(a) meets the curve again in the third quadrant
(b) meets the curve again in the fourth quadrant
(c) does not meet the curve again
(d) meets the curve again in the second quadrant

35. The complete set of value of 'a' for which there exists atleast one line that is tangent to the graph of the curve $y = x^3 - ax$ at one point and normal to the graph at another point is given by
(a) $\left(-\infty, -\frac{4}{3}\right]$ (b) $\left[-\frac{4}{3}, \infty\right)$ (c) $\left[\frac{4}{3}, \infty\right)$ (d) $\left(-\infty, \frac{4}{3}\right]$

Answers

1. (b)	2. (d)	3. (d)	4. (a)	5. (b)
6. (b)	7. (a)	8. (b)	9. (c)	10. (d)
11. (a)	12. (a)	13. (c)	14. (d)	15. (d)
16. (a)	17. (b)	18. (d)	19. (b)	20. (a)
21. (a)	22. (c)	23. (b)	24. (a)	25. (a)
26. (b)	27. (b)	28. (a)	29. (c)	30. (c)
31. (b)	32. (a)	33. (c)	34. (b)	35. (c)

INCREASING AND DECREASING FUNCTIONS (MONOTONOCITY)

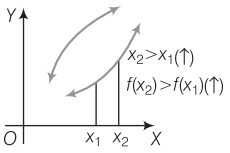
Monotonocity is the most important behaviour of the functions. The word monotonocity means increasing (ascending) or decreasing (descending). If function output is increasing on increasing input in its entire domain, then the function is said to be an increasing function and if output of the function decreases on increasing input, then function is called decreasing function.

Strictly Increasing Function

A function $f(x)$ is known as strictly increasing function in its domain, if
$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

or

If a function is strictly increasing in $[a, b]$, then $f'(x) > 0, \forall\, x \in [a, b]$



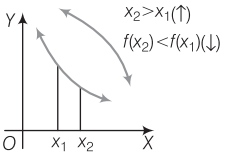
Strictly increasing function

Strictly Decreasing Function

A function $f(x)$ is said to be strictly decreasing function in its domain, if
$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

or

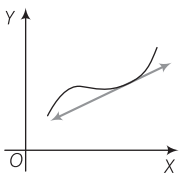
If the function is strictly decreasing in $[a, b]$, then $f'(x) < 0, \forall\, x \in [a, b]$.



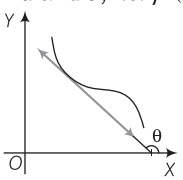
Strictly decreasing function

Geometrical Interpretation

(i) If $f(x)$ is strictly increasing in $[a, b]$, then tangent to curve has a positive slope at all points which lies between a and b, i.e. slope of tangent line will be acute.

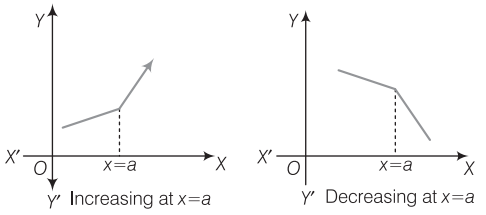


(ii) If $f(x)$ is strictly decreasing in $[a, b]$, then tangent to curve has negative slope at all points of the curve lying in between a and b, i.e. $f'(x) < 0, \forall\, x \in [a, b]$.

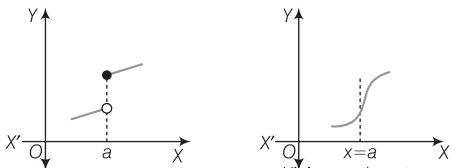


Note If $f'(x) \geq 0, \forall\ x \in [a, b]$, then $f(x)$ is simply called an increasing function in $[a, b]$. Similarly, if $f'(x) \leq 0, \forall\ x \in [a, b]$, then $f(x)$ is simply called a decreasing function in $[a, b]$.

We can take of monotonocity of $f(x)$ at $x = a$ only if $x = a$ lies in the domain of f, without any consideration of continuity or differentiability of $f(x)$ at $x = a$.



(Not differentiable at $x=a$)

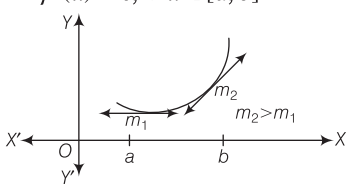


Concavity of Curves

A curve is called concave downward, if all points of the curve lie below the tangent to the curve at any point and curve is called concave upward, if all the points lie above the tangent drawn to the curve at any point. Geometrically, it can be understand in the following ways.

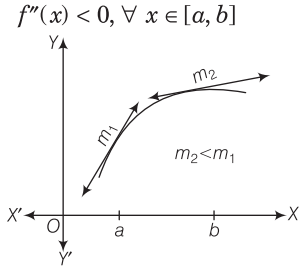
(i) If the curve is concave upwards, its slope increases as x increases, i.e. $f'(x)$ is strictly increasing in $[a, b]$.

$\Rightarrow$ $f''(x) > 0, \forall\ x \in [a, b]$



(ii) If the curve is concave downwards, then the slope of the tangent decreases as x increases, i.e. $f'(x)$ is strictly decreasing in $[a, b]$.

$\Rightarrow$ $f''(x) < 0, \forall\ x \in [a, b]$



Monotonocity in an Interval

Let I be an open interval contained in the domain of real valued function f.

(i) Function f is increases on I, if $x_1 < x_2$ in I.

$\Rightarrow$ $f(x_1) \leq f(x_2), \forall\ x_1$ and $x_2 \in I$

(ii) Function f is decreasing on I, if $x_1 < x_2$ in I.

$\Rightarrow$ $f(x_1) \geq f(x_2), \forall\ x_1$ and $x_2 \in I$

(iii) Function f is strictly increasing on I, if $x_1 < x_2$ in I.

$\Rightarrow$ $f(x_1) < f(x_2), \forall\ x_1 x_2 \in I$

Point of Inflection

A point on the curve, where the curve changes its nature of concavity is called point of inflection.

If f having derivative of all order at $x = a$, such that $f'(a) = 0 = f''(a)$ and $f'''(a) \neq 0$, then the point $x = a$ is known as point of inflection.

Note Monotonic function have atmost one root.

Properties of Monotonic Functions

(i) If $f(x)$ is strictly increasing function on an interval $[a, b]$, then f^{-1} exists and also a strictly increasing function.

(ii) If $f(x)$ is strictly increasing function on $[a, b]$ such that it is continuous, then f^{-1} is continuous on $[f(a), f(b)]$.

(iii) If $f(x)$ and $g(x)$ are strictly increasing (or decreasing) function on $[a, b]$, then $gof(x)$ is strictly increasing (or decreasing) function on $[a, b]$.

(iv) If one of the two functions $f(x)$ and $g(x)$ is strictly increasing and other is strictly decreasing, then $gof(x)$ is strictly decreasing on $[a, b]$.

(v) If $f(x)$ is continuous on $[a, b]$ such that $f'(c) > 0$ for each $c \in (a, b)$, then $f(x)$ is strictly increasing function on $[a, b]$.

(vi) If $f(x)$ is continuous on $[a, b]$ such that $f'(c) < 0$ for each $c \in (a, b)$, then $f(x)$ is strictly decreasing function on $[a, b]$.

The Amazing Four

There are just four numbers (after 1) which are the sums of the cubes of their digits :

$$153 = (1\wedge 3)+(5\wedge 3)+(3\wedge 3)$$

$$370 = (3\wedge 3)+(7\wedge 3)+(0\wedge 3)$$

$$371 = (3\wedge 3)+(7\wedge 3)+(1\wedge 3)$$

$$407 = (4\wedge 3)+(0\wedge 3)+(7\wedge 3)$$

[$\wedge$ means raised to power]

MASTER STROKES

1. The function f defined by $f(x) = (x+2)e^{-x}$ is
(a) decreasing for all x
(b) decreasing in $(-\infty, -1)$ and increasing in $(-1, \infty)$
(c) increasing for all x
(d) decreasing in $(-1, \infty)$ and increasing in $(-\infty, -1)$

2. Let $f(x) = -x^3 + ax^2 + bx + 5\sin x \cos x$ be a decreasing function on the set R. Then, a and b satisfy
(a) $a^2 + 3b + 15 > 0$ (b) $a^2 + 3b + 15 < 0$
(c) $a^2 - 3b - 15 > 0$ (d) $a^2 - 3b - 15 < 0$

3. Let $h(x) = f(x) - \{f(x)\}^2 + \{f(x)\}^3$ for every real number x. Then,
(a) h is increasing whenever f is increasing
(b) h is increasing whenever f is decreasing
(c) h is decreasing whenever f is increasing
(d) nothing can be said in general

4. If $f(x) = \dfrac{x}{\sin x}$ and $g(x) = \dfrac{x}{\tan x}$, where $0 < x \le 1$, then in this interval,
(a) both $f(x)$ and $g(x)$ are increasing functions
(b) both $f(x)$ and $g(x)$ are decreasing functions
(c) $f(x)$ is an increasing function
(d) $g(x)$ is an increasing function

5. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in
(a) $\left(0, \dfrac{\pi}{2}\right)$ (b) $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
(c) $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$ (d) $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{4}\right)$

6. The set of all x for which $1 + \log x < x$, is
(a) $(1, \infty)$ (b) $(0, 1)$
(c) $(0, \infty)$ (d) None of these

7. If the function $f(x) = 2x^2 - kx + 5$ is increasing on $[1, 2]$, then k lies in the interval
(a) $(-\infty, 4)$ (b) $(4, \infty)$
(c) $(-\infty, 8)$ (d) $(8, \infty)$

8. If a, b, c are real, then $f(x) = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$ is decreasing on
(a) $\left(\dfrac{-2}{3}(a^2+b^2+c^2), 0\right)$
(b) $\left(0, \dfrac{2}{3}(a^2+b^2+c^2)\right)$
(c) $\left(\dfrac{a^2+b^2+c^2}{3}, 0\right)$
(d) None of the above

9. If the function $f(x) = 2\tan x + (2a+1)\log_e |\sec x| + (a-2)x$ is increasing on R, then
(a) $a \in \left(\dfrac{1}{2}, \infty\right)$ (b) $a \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$
(c) $a = \dfrac{1}{2}$ (d) $a \in R$

10. If $f'(x^2 - 4x + 3) > 0$ for all $x \in (2, 3)$, then $f(\sin x)$ is increasing on
(a) $\bigcup_{n \in Z} \left(2n\pi, (4n+1)\dfrac{\pi}{2}\right)$ (b) $\bigcup_{n \in Z} \left((4n-1)\dfrac{\pi}{2}, 2n\pi\right)$
(c) R (d) None of these

11. Let $f(x) = \tan^{-1}(g(x))$, where $g(x)$ is monotonically increasing for $0 < x < \dfrac{\pi}{2}$. Then, $f(x)$ is
(a) increasing on $\left(0, \dfrac{\pi}{2}\right)$
(b) decreasing on $\left(0, \dfrac{\pi}{2}\right)$
(c) increasing on $\left(0, \dfrac{\pi}{4}\right)$ and decreasing on $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$
(d) None of the above

12. Let $f(x) = \int e^x (x-1)(x-2)dx$, then $f(x)$ decreases in the interval
(a) $(-\infty, -2)$ (b) $(-2, -1)$
(c) $(1, 2)$ (d) $(2, \infty)$

13. The length of the longest interval in which the function $3\sin x - 4\sin^3 x$ is increasing, is
(a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{2}$ (c) $\dfrac{3\pi}{2}$ (d) π

14. Let $f'(x) > 0$ and $g'(x) < 0$ for all $x \in R$. Then,
(a) $f\{g(x)\} < f\{g(x+1)\}$ (b) $f\{g(x)\} > f\{g(x-1)\}$
(c) $g\{f(x)\} > g\{f(x+1)\}$ (d) $g\{f(x)\} > g\{f(x-1)\}$

15. If $f(x) = (ab + b^2 + 1)x + \int_0^x (\cos^4\theta + \sin^4\theta)\, d\theta$ is an increasing function of x for all $x \in R$ and $b \in R$, b being independent of x, then
(a) $a \in (0, \sqrt{6})$ (b) $a \in (-\sqrt{6}, \sqrt{6})$
(c) $a \in (-\sqrt{6}, 0)$ (d) None of these

16. If $f(x) = \dfrac{a^2-1}{a^2+1}x^3 - 3x + 2\log_e 5$ is a decreasing function of x for all $x \in R$, then the set of possible values of a is
(a) $[-1, 1]$ (b) $[1, \infty)$ (c) $(-\infty, -1]$ (d) $(-\infty, \infty)$

17. Let $f(x) = \begin{cases} xe^{ax}, & x \le 0 \\ x + ax^2 - x^3, & x > 0 \end{cases}$, where a is a positive constant. Then, the interval in which $f'(x)$ is increasing, is

(a) $\left(0, \frac{a}{3}\right)$ (b) $\left(-\frac{2}{a}, 0\right)$

(c) $\left(-\frac{2}{a}, \frac{a}{3}\right)$ (d) None of these

18. Let $f(x)$ be a function given by $f(x) = \frac{x^2+1}{[x]}$ for all $x \in [1, 4)$, where $[\cdot]$ denotes the greatest integer function. Then, $f(x)$ is monotonically

(a) increasing on $[1, 4)$
(b) decreasing on $[1, 4)$
(c) increasing on $[1, 2)$
(d) decreasing on $[2, 3)$

19. If $0 < \alpha < \beta < \frac{\pi}{2}$, then

(a) $\frac{\tan\beta}{\tan\alpha} < \frac{\alpha}{\beta}$ (b) $\frac{\tan\beta}{\tan\alpha} > \frac{\alpha}{\beta}$

(c) $\frac{\tan\alpha}{\tan\beta} < \frac{\alpha}{\beta}$ (d) $\frac{\tan\alpha}{\tan\beta} > \frac{\alpha}{\beta}$

20. If $f(x) = xe^{x(1-x)}$, then $f(x)$ is

(a) increasing on $\left[-\frac{1}{2}, 1\right]$ (b) decreasing on R

(c) increasing on R (d) decreasing on $\left[-\frac{1}{2}, 1\right]$

21. If $0 < x < \frac{\pi}{2}$, then

(a) $\cos x > 1 - \frac{2x}{\pi}$ (b) $\cos x < 1 - \frac{2x}{\pi}$

(c) $\cos x > \frac{2x}{\pi}$ (d) $\cos x < \frac{2x}{\pi}$

22. Let $f'(\sin x) < 0$ and $f''(\sin x) > 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$ and $g(x) = f(\sin x) + f(\cos x)$, then $g(x)$ is decreasing in

(a) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (b) $\left(0, \frac{\pi}{4}\right)$

(c) $\left(0, \frac{\pi}{2}\right)$ (d) $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$

23. Let $f(x) = 1 - x - x^3$, then the real values of x satisfying the inequality $1 - f(x) - f^3(x) > f(1 - 5x)$ is

(a) $(0, 2)$ (b) $(-2, 2)$

(c) $(-2, 1) \cup (1, \infty)$ (d) $(-\infty, -2) \cup (0, 2)$

24. The interval in which the function $f(x) = \int_0^x \left(\frac{t}{t+2} - \frac{1}{t}\right) dt$ will be non-increasing, is

(a) $(-2, -1] \cup (0, 3]$ (b) $(-2, -1] \cup [0, 3]$

(c) $(-2, -1] \cup [0, 2]$ (d) $(-2, -1] \cup (0, 2]$

25. Let $f(x)$ be a monotic polynomial of degree $(2m - 1)$, where $m \in N$. Then, the equation $f(x) + f(3x) + f(5x) + \ldots + f((2m-1)x) = (2m-1)$ has

(a) atleast one real root (b) $(2m - 1)$ real roots

(c) exactly one real root (d) None of these

26. If $f(x) = \int_x^{x^2} \frac{1}{(\log t)^2} dt$, $x \neq 0$, $x \neq 1$, then $f(x)$ is monotonically

(a) increasing on $(2, \infty)$
(b) increasing on $(1, 2)$
(c) decreasing on $(2, \infty)$
(d) decreasing on $(0, 3)$

27. $f(x) = |x \log_e x|$, $x > 0$, is monotonically decreasing in

(a) (e, ∞) (b) $\left(0, \frac{1}{e}\right)$

(c) $\left[\frac{1}{e}, 1\right]$ (d) $(1, e)$

28. If $f(x) = x^3 + bx^2 + cx + d$ and $0 < b^2 < c$, then in $(-\infty, \infty)$,

(a) $f(x)$ is strictly increasing function
(b) $f(x)$ has a local maxima
(c) $f(x)$ is a strictly decreasing function
(d) $f(x)$ is unbounded

29. Consider the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3$. Let S be the sum of all distinct real roots of $f(x)$. Then, S lies in the interval

(a) $\left(-\frac{1}{4}, 0\right)$ (b) $\left(-11, -\frac{3}{4}\right)$

(c) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (d) $\left(0, \frac{1}{4}\right)$

30. Consider the function $f(x) = 1 + 2x + 3x^2 + 4x^3$. Let S be the sum of all distinct real roots of $f(x)$ and let $t = |S|$. The function $f'(x)$ is

(a) increasing in $\left(-t, -\frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$

(b) decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$

(c) increasing in $(-t, t)$
(d) decreasing in $(-t, t)$

31. Let the function $g : (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is

(a) even and is strictly increasing in $(0, \infty)$
(b) odd and is strictly decreasing in $(-\infty, \infty)$
(c) odd and is strictly increasing in $(-\infty, \infty)$
(d) neither even nor odd, but is strictly increasing in $(-\infty, \infty)$

32. For the function $f(x) = x\cos\frac{1}{x}$, $x \geq 1$, which one of the following is incorrect?

(a) for atleast one x in the interval $[1, \infty)$, $f(x + 2) - f(x) < 2$
(b) $\lim_{x \to \infty} f'(x) = 1$
(c) for all x in the interval $[1, \infty)$, $f(x + 2) - f(x) > 2$
(d) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$

33. If $f(x) = x^{3/2}(3x - 10)$, $x \geq 0$, then $f(x)$ is increasing in

(a) $(-\infty, -1) \cup (1, \infty)$ (b) $[2, \infty)$
(c) $(-\infty, -1) \cup [1, \infty)$ (d) $(-\infty, 0] \cup (2, \infty)$

34. Let $f(x) = (1-x)^2 \sin^2 x + x^2$ for all $x \in R$ and let

$$g(x) = \int_1^x \left\{ \frac{2(t-1)}{t+1} - \log_e t \right\} f(t)\, dt \quad \text{for all } x \in (1, \infty).$$

Then, which of the following is true?

(a) g is increasing on $(1, \infty)$
(b) g is decreasing on $(1, \infty)$
(c) g is increasing on $(1, 2)$ and decreasing on $(2, \infty)$
(d) g is decreasing on $(1, 2)$ and increasing on $(2, \infty)$

35. Consider the function $f : R \to R$ given by

$$f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}, 0 < a < 2.$$

If $g(x) = \int_0^{e^x} \frac{f'(t)}{1+t^2}\, dt$, which of the following is true?

(a) $g'(x)$ is positive on $(-\infty, 0)$ and negative on $(0, \infty)$
(b) $g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$
(c) $g'(x)$ changes sign on both $(-\infty, 0)$ and $(0, \infty)$
(d) $g'(x)$ does not change sign on $(-\infty, \infty)$

Answers

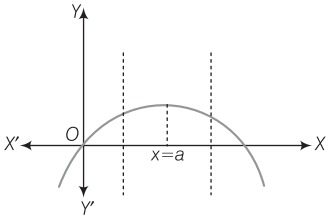
1. (d)	2. (b)	3. (a)	4. (c)	5. (d)
6. (c)	7. (a)	8. (a)	9. (c)	10. (a)
11. (a)	12. (c)	13. (a)	14. (c)	15. (b)
16. (a)	17. (c)	18. (c)	19. (b)	20. (a)
21. (a)	22. (b)	23. (d)	24. (d)	25. (c)
26. (a)	27. (c)	28. (a)	29. (c)	30. (b)
31. (c)	32. (a)	33. (b)	34. (b)	35. (b)

MAXIMA AND MINIMA

The maximum/minimum value of a function $f(x)$ implies local or regional maximum/minimum and not the greatest/least value attained by the function. It is also possible in a function that local maximum at one point is smaller than local minimum at another point. Sometimes we use the word extrema for the maxima and minima.

Maximum

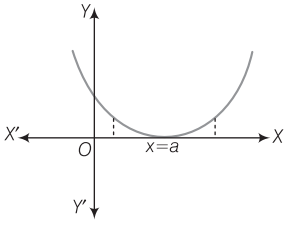
A function $f(x)$ is said to have a maximum or local maximum at $x = a$, if the value of $f(a)$ is greater than all the values of $f(x)$ in a small neighbourhood of $x = a$.



Maximum at $x = a$, $f(a-h) < f(a) > f(a+h)$

Minimum

A function $f(x)$ is said to have minimum at $x = a$, if the value of $f(a)$ is less than all values of $f(x)$ in a small neighbourhood of $x = a$.



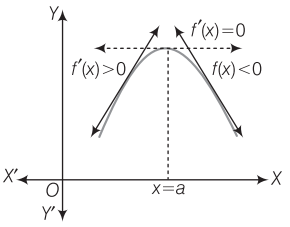
Minimum at $x = 0$, $f(a-h) > f(a) < f(a+h)$

Methods to Find Extremum of a Function

(i) First Derivative Test

Maxima

A function $f(x)$ is said to be maximum at $x = a$, if nature of the function changes from increasing to decreasing i.e. $f'(x)$ changes sign from positive to negative at $x = a$ and $f'(a) = 0$.

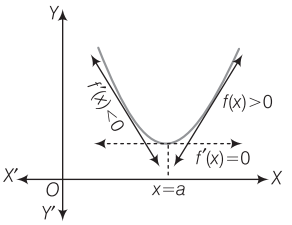


Thus, $f'(x)$ decreases from positive to zero and then negative.

Hence, $f''(x) < 0$ i.e. $f''(x)$ is negative.

Minima

The function $f(x)$ is said to be minimum at $x = a$, if nature of the function changes from decreasing to increasing as we move from left to right in the neighbourhood of x, i.e. $f'(x)$ changes sign from negative to positive. Thus, $f'(x)$ increases from negative to zero and then positive. Hence, $f''(x) > 0$ i.e. $f''(x)$ is positive.



(ii) Second Derivative Test

To find the extremum of a function $f(x)$, we use following steps

***Step* I** Find $f'(x)$ and equate it to zero i.e. $f'(x) = 0$.

***Step* II** Solve $f'(x) = 0$, let its roots (critical points) be $x = a_1, a_2, \ldots$.

***Step* III** Find $f''(x)$ at $x = a_1$

(a) If $f''(a_1)$ is positive, then $f(x)$ is minimum at $x = a_1$.

(b) If $f''(a_1)$ is negative, then $f(x)$ is maximum at $x = a_1$.

***Step* IV** If at $x = a_1$, $f''(a_1) = 0$, then find $f'''(x)$. If $f'''(a_1) \neq 0$. Then, $f(x)$ is neither maximum nor minimum at $x = a_1$.

***Step* V** If $f'''(a_1) = 0$, then find $f^{iv}(x)$

(a) If $f^{iv}(a_1)$ is positive, then $f(x)$ is minimum at $x = a_1$.

(b) If $f^{iv}(a_1)$ is negative, then $f(x)$ is maximum at $x = a_1$.

***Step* VI** If at $x = a_1$, $f^{iv}(a_1) = 0$, then find $f^{v}(x)$ proceed same as step IV and step V. Similarly, check for the other values of x i.e. $a_2, a_3, \ldots$.

Concept of Absolute Global Maxima/Minima

Let $y = f(x)$ be a given function with domain D.

Let $[a, b] \subseteq D$, then maximum/minimum of $f(x)$ in $[a, b]$ is basically the greatest and least value of $f(x)$ in $[a, b]$. Absolute maxima/minima in $[a, b]$ would always occurs at critical points of $f(x)$ within $[a, b]$ or at the end point of the interval.

Absolute Maximum or Minimum Value of a Function

In order to find absolute maximum or minimum value of a function, we use the following steps.

***Step* I** Check the continuity of the function over $[a, b]$.

***Step* II** List end points or critical values i.e. $a, c_1, c_2, \ldots, b$.

***Step* III** Evaluate $f(a), f(c_1), f(c_2), \ldots, f(b)$.

***Step* IV** The absolute maximum value is the greatest value found in step III and the absolute minimum value is smallest values found in step III.

SPEED BOOSTER

(i) The function $y = \ln f(x)$ is maximum or minimum according as $f(x)$ is maximum or minimum.

(ii) The function $y = kf(x)$ or $y = \{f(x)\}^k$ $(k > 0)$ is maximum or minimum according as $f(x)$ is maximum or minimum.

(iii) If $a_1, a_2, a_3, \ldots, a_n = k$ (constant), then the value of $a_1 + a_2 + a_3 + \ldots + a_n$ is least, when $a_1 = a_2 = a_3 = \ldots = a_n$ Hence, least value of $(a_1 + a_2 + a_3 + \ldots + a_n) = n(k)^{1/n}$.

(iv) If $a_1^m + a_2^m + a_3^m + \ldots + a_n^m = k$ and $m \notin (0, 1)$ and the greatest value of $a_1 + a_2 + a_3 + \ldots + a_n$ occurs when $a_1 = a_2 = a_3 = \ldots = a_n$, then the value is $n^{1-\frac{1}{m}} \cdot k^{1/m}$.

(v) $y = f(x)$ is maximum or minimum as $\frac{1}{f(x)}$ is maximum or minimum.

(vi) The function $f(x) = \frac{ax + b}{cx + d}$ has no local maximum or minimum regardless of the values of a, b, c and d.

(vii) If the perimeter of rectangles is given, then square will have largest area.

(viii) If area of rectangle is given, then square has the least perimeter.

(ix) $f(x) = \sin^m x \cos^n x$ attains maximum value at $x = \tan^{-1}\sqrt{m/n}$.

(x) A cone of maximum volume can be inscribed in a sphere of given radius (r) of height $4r/3$ and right circular cylinder of height $2r/\sqrt{3}$.

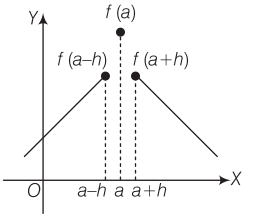
Maxima and Minima of Discontinuous Function

For discontinuous function, we don't apply the methods discussed earlier. We observe certain conditions and their graphs which would give you more clear picture.

(i) Maximum of Discontinuous Functions

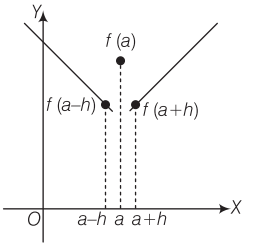
For the maxima of discontinuous functions, we have following four cases

***Case* I**



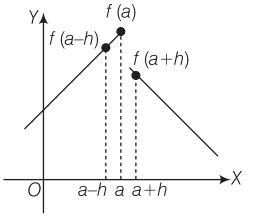
From this figure, $f(a) > f(a + h)$, $f(a) > f(a - h)$

***Case* II**



From this figure, $f(a) > f(a + h)$, $f(a) > f(a - h)$

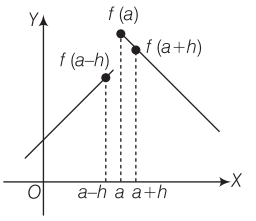
***Case* III**



From this figure,

$$f(a) > f(a + h), \quad f(a) \geq f(a - h)$$

***Case* IV**



From this figure, $f(a) \geq f(a + h)$, $f(a) > f(a - h)$

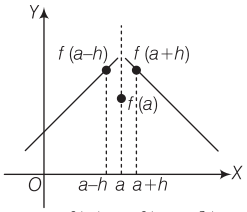
From all the above four mentioned cases for maximum of discontinuous functions, we have

$$f(a) \geq f(a + h), f(a) \geq f(a - h)$$

(ii) Minimum of Discontinuous Functions

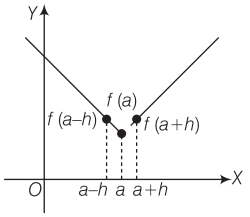
For the minimum of discontinuous functions, the following four cases arise at $x = a$.

***Case* I**



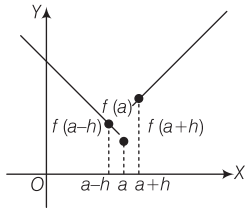
From this figure, $f(a) < f(a + h)$, $f(a) < f(a - h)$

***Case* II**



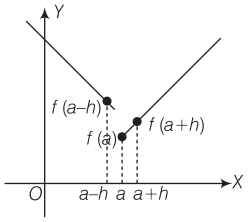
From this figure, $f(a) < f(a + h)$, $f(a) \le f(a - h)$

***Case* III**



From this figure, $f(a) < f(a + h)$, $f(a) < f(a - h)$

***Case* IV**



From this figure, $f(a) \le f(a + h)$, $f(a) < f(a - h)$

From all the above four mentioned cases for minimum of discontinuous functions, we have

$f(a) \le f(a + h)$ and $f(a) \le f(a - h)$

Properties of Maxima and Minima

(i) If $f(x)$ is continuous function in its domain, then atleast one maxima and one minima must lie between two equal values of x.

(ii) Maxima and minima occur alternatively i.e. between two maxima, there is one minima and *vice-versa*.

(iii) If $f(x) \to \infty$ as $x \to a$ or b and $f'(x) = 0$ only for one value of x (say $c \in (a, b)$), then $f(c)$ is necessarily the minimum and the least value.

(iv) If $f(x) \to -\infty$ as $x \to a$ or b, then $f(c)$ is necessarily the maximum and the greatest value.

MASTER STROKES

1. If $f(x) = \int_0^x (t^2 + 2t + 2)\, dt$, where $x \in [2, 4]$, then

(a) the maximum value of $f(x)$ is $\frac{32}{3}$

(b) the minimum value of $f(x)$ is 10

(c) the maximum value of $f(x)$ is 26

(d) None of the above

2. On the interval [0, 1], the function $x^{25}(1 - x)^{75}$ takes its maximum value at the point

(a) 0 *(b)* 1/4 *(c)* 1/2 *(d)* 1/3

3. The value of a so that the sum of the cubes of the roots of the equation $x^2 - ax + (2a - 3) = 0$ assumes the minimum value, is

(a) $a = 1$ *(b)* $a = 3$

(c) $a = 0$ *(d)* $a = 2$

4. The minimum value of the function $f(x) = \frac{a^2}{x} + \frac{b^2}{a - x}$, $a > 0$, $b > 0$, in $(0, a)$ is

(a) $a + b$ *(b)* $\frac{1}{a + b}$

(c) $\frac{(a + b)^2}{a}$ *(d)* $\frac{a + b}{a^2}$

5. The minimum value of $a \tan^2\theta + b \cot^2\theta$ equals the maximum of $a \sin^2\theta + b \cos^2\theta$, where $a > b > 0$, when

(a) $a = b$ *(b)* $a = 2b$

(c) $a = 3b$ *(d)* $a = 4b$

6. Let $f(x) = \cos \pi x + 10x + 3x^2 + x^3$, $x \in [-2, 3]$. The absolute minimum value of $f(x)$ is

(a) 0 *(b)* -15 *(c)* $3 - 2\pi$ *(d)* $3 + 2\pi$

7. If $f(x) = a \log|x| + bx^2 + x$ has its extremum values at $x = -1$ and $x = 2$, then

(a) $a = 2, b = -1$ *(b)* $a = 2, b = -1/2$

(c) $a = -2, b = 1/2$ *(d)* None of these

8. The critical points of $f(x) = \frac{|2 - x|}{x^2}$ is/are

(a) $x = 0, 2$ *(b)* $x = 2, 4$

(c) $x = 2, -4$ *(d)* None of these

9. The value of a for which the function $f(x) = (4a - 3)(x + \log 5) + 2(a - 7)\cot\frac{x}{2}\sin^2\frac{x}{2}$ does not posses critical points, is

(a) $(-\infty, -4/3)$ *(b)* $(-\infty, -1)$

(c) $[1, \infty)$ *(d)* $(-2, 2)$

10. Suppose the cubic $x^3 - px + q$ has three real roots where $p > 0$ and $q > 0$. Then, which one of the following holds?

(a) The cubic has minimum at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

(b) The cubic has maximum at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

(c) The cubic has minimum at $\sqrt{\frac{p}{3}}$ and maximum at $-\sqrt{\frac{p}{3}}$

(d) The cubic has minimum at $-\sqrt{\frac{p}{3}}$ and maximum at $\sqrt{\frac{p}{3}}$

11. If p and q are positive real numbers such that $p^2 + q^2 = 1$, then the maximum value of $p + q$, is

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$

(c) 2 (d) $\frac{1}{2}$

12. If $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$ and $P(-1) < P(1)$, then in the interval $[-1, 1]$,

(a) $P(-1)$ is the maximum and $P(1)$ the maximum of P

(b) $P(-1)$ is not minimum but $P(1)$ the maximum of P

(c) $P(-1)$ is the minimum but $P(1)$ is not the maximum of P

(d) Neither $P(-1)$ is the maximum nor $P(1)$ is the maximum of P

13. A straight line through the point (h, k) where $h > 0$ and $k > 0$, makes positive intercepts on the coordinates axes. Then, the minimum length of the line intercepted between the coordinate axes, is

(a) $(h^{2/3} + k^{2/3})^{3/2}$ (b) $(h^{3/2} + k^{3/2})^{2/3}$

(c) $(h^{2/3} - k^{2/3})^{3/2}$ (d) $(h^{3/2} - k^{3/2})^{2/3}$

14. The total number of local maxima and local minima for the function

$$f(x) = \begin{cases} (2+x)^3, & -3 < x \le -1 \\ x^{2/3}, & -1 < x < 2 \end{cases}$$ is

(a) 0 (b) 1

(c) 2 (d) 3

15. If $f : R \to R$ is defined by $f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}$, $0 < a < 2$, then which of the following is true?

(a) $f(x)$ is decreasing on $(-1, 1)$ and has a local minimum at $x = 1$

(b) $f(x)$ is increasing on $(-1, 1)$ and has a local maximum at $x = 1$

(c) $f(x)$ is increasing on $(-1, 1)$ and has neither a local maximum nor a local minimum at $x = 1$

(d) $f(x)$ is decreasing on $(-1, 1)$ but has neither a local maximum nor a local minimum at $x = 1$

16. If $f(x)$ is a cubic polynomial which has local maximum at $x = -1$. If $f(2) = 18$, $f(1) = -1$ and $f'(x)$ has local minimum at $x = 0$, then

(a) the distance between $(-1, 2)$ and $(\alpha, f(\alpha))$, where $x = \alpha$ is the point of local minima, is $2\sqrt{5}$

(b) $f(x)$ is increasing for $x \in [1, 2\sqrt{5}]$ and has a local minima at $x = 1$

(c) the value of $f(0)$ is 5

(d) None of the above

17. Let $f(x) = \int_0^x (\sin t - \cot t)\,(e^t - 2)\,(t-1)^3(t-2)^5\,dt$, $0 < x \le 4$. Then, the number of points where $f(x)$ assumes local maximum value, is

(a) 1 (b) 2

(c) 3 (d) 4

18. Let $f(x) = (x-2)^2 x^n$, $n \in N$. Then, $f(x)$ has a minimum at

(a) $x = 2$ for all $x \in N$ (b) $x = 2$, if n is even

(c) $x = 0$, if n is odd (d) $x = 0$, if n is odd

19. The function $f(x) = \frac{x}{1 + x \tan x}$ has

(a) one point of minimum in the interval $(0, \pi/2)$

(b) one point of maximum in the interval $(0, \pi/2)$

(c) no point of maximum, no point of minimum in the interval $(0, \pi/2)$

(d) two points of maxima in the interval $(0, \pi/2)$

20. The fraction exceeding its nth power by the greatest possible number, where $n \ge 2$, is

(a) $\left(\frac{1}{n}\right)^{\frac{1}{n-1}}$ (b) $\left(\frac{1}{n}\right)^{n-1}$ (c) $n^{\frac{1}{n}-1}$ (d) $\left(\frac{1}{n} - 1\right)^n$

21. The minimum value of the function $f(x) = 2|x-2| + 5|x-3|$ for all $x \in R$, is

(a) 3 (b) 4 (c) 5 (d) 2

22. The minimum value of the function $f(x)$ given by $f(x) = \frac{x^m}{m} + \frac{x^{-n}}{n}$, where $\frac{1}{m} + \frac{1}{n} = 1$ and $m > 1$, is

(a) 3 (b) 2 (c) 1 (d) 0

23. The largest term of the sequence $< a_n >$ given by $a_n = \frac{n^2}{n^3 + 200}$, $n \in N$ is

(a) $\frac{49}{543}$ (b) $\frac{8}{89}$

(c) $\frac{1}{52}$ (d) None of these

24. If $P = x^3 - \frac{1}{x^3}$ and $Q = x - \frac{1}{x}$, $x \in (0, \infty)$, then minimum value of $\frac{P}{Q^2}$ is

(a) $2\sqrt{3}$ (b) $-2\sqrt{3}$

(c) non-existent (d) 0

25. If the parabola $y = ax^2 + bx + c$ has vertex at $(4, 2)$ and $a \in [1, 3]$, then the difference between the extreme values of abc is equal to

(a) 3600 (b) 144 (c) 3456 (d) 441

26. Let f, g, h be real valued functions defined on the interval $[0, 1]$ by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$. If a, b and c denote respectively, the absolute maximum of f, g and h on $[0, 1]$, then

(a) $a = b$ and $c \neq b$ (b) $a = c$ and $a \neq b$
(c) $a \neq b$ and $c \neq b$ (d) $a = b = c$

27. Let f be a function defined on R (the set of all real numbers) such that $f'(x) = 2010\,(x - 2009)\,(x - 2010)^2\,(x - 2011)^3\,(x - 2012)^4$, for all $x \in R$. If g is a function defined on R with values in the interval $(0, \infty)$ such that $f(x) = \ln\{g(x)\}$ for all $x \in R$, then the number of points in R at which g has a local maximum, is

(a) 1 (b) 2
(c) 3 (d) 4

28. Let $f : R \to R$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum, is

(a) 2 (b) 4
(c) 5 (d) 6

29. Let $p(x)$ be a real polynomial of least degree which has a local maximum at $x = 1$ and a local minimum at $x = 3$. If $p(1) = 6$ and $p(3) = 2$, then $p'(0)$ is

(a) 8 (b) 9 (c) 3 (d) 6

30. Let $f : [0, 1] \to R$ be a function. Suppose the function f is twice differentiable with $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x$ for all $x \in [0, 1]$. Which of the following is true for $x \in (0, 1)$?

(a) $0 < f(x) < \infty$ (b) $-\frac{1}{2} < f(x) < \frac{1}{2}$
(c) $-\frac{1}{4} < f(x) < 1$ (d) $-\infty < f(x) < 0$

31. A cylindrical container is to be made from certain solid material with the following constraints : It has a fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container. If the volume of the material used to make the container is minimum when the inner radius of the container is 10 mm, then the value of $\frac{V}{250\,\pi}$ is

(a) 6 (b) 8
(c) 7 (d) 4

32. Let $f(x)$ be a polynomial of degree four having extreme values at $x = 1$ and $x = 2$. If $\lim\limits_{x \to 0}\left(1 + \frac{f(x)}{x^2}\right) = 3$, then $f(2)$ is equal to

(a) 0 (b) 0 (c) -8 (d) -4

33. The number of points in the interval $[-\sqrt{13}, \sqrt{13}]$ at which $f(x) = \sin x^2 + \cos x^2$ attains its maximum value, is

(a) 2 (b) 8 (c) 0 (d) 4

34. Let $f(x) = |x - x_1| + |x - x_2|$, where x_1 and x_2 are distinct real numbers. Then, the number of points at which $f(x)$ is minimum, is

(a) more than 3 (b) 1
(c) 2 (d) 3

35. The function $f(x) = 2|x| + |x + 2| - |x + 2| - 2|x|$ has a local minima or a local maxima respectively at x equal to

(a) -2 and $-\frac{2}{3}$ (b) -2 and 0
(c) $-\frac{2}{3}$ and 2 (d) 2 and -2

36. The minimum value of the function $f(x) = x^{3/2} + x^{-3/2} - 4\left(x + \frac{1}{x}\right)$ for all permissible real x is

(a) -10 (b) -6
(c) -7 (d) -8

37. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed square is 100, the resulting box has maximum value, the dimensions of the sides of the rectangular sheet are

(a) 24, 45 (b) 32, 65 (c) 24, 60 (d) 32, 60

Answers

1. (a)	**2.** (b)	**3.** (b)	**4.** (c)	**5.** (d)
6. (b)	**7.** (b)	**8.** (d)	**9.** (a)	**10.** (c)
11. (b)	**12.** (b)	**13.** (a)	**14.** (c)	**15.** (a)
16. (b)	**17.** (c)	**18.** (a)	**19.** (b)	**20.** (a)
21. (d)	**22.** (c)	**23.** (a)	**24.** (a)	**25.** (c)
26. (d)	**27.** (a)	**28.** (c)	**29.** (b)	**30.** (d)
31. (d)	**32.** (a)	**33.** (d)	**34.** (a)	**35.** (a)
36. (a)	**37.** (a)			

Brain Shaker!!

A cylindrical container has a radius of eight inches with a height of three inches. Compute how many inches should be added to either the radius or height to give the same increase in volume?

Er. Love Agrawal

JEE ADVANCED DRILL

A collection of 30 innovative problems on JEE Advanced pattern from complete syllabus

Single Option Correct Type

1. The tangent at the point α on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the auxiliary circle in two points which subtend a right angle at the centre, the eccentricity of the ellipse, is

(a) $\sqrt{1+\sin^2\alpha}$ *(b)* $\frac{1}{\sqrt{1+\sin^2\alpha}}$

(c) $\frac{1}{\sqrt{1+\cos^2\alpha}}$ *(d)* $\sqrt{1+\cos^2\alpha}$

2. If $g(x)$ is a quadratic expression positive for all real numbers x and $h(x)=g(x)-g'(x)+g''(x)$, then

(a) $h(x)\geq 0$ *(b)* $h(x)\leq 0$

(c) $h(x)>0$ *(d)* $h(x)<0$

3. Let $a_i\in R; i=1, 2, 3, \ldots, n$ and all a_i's are distinct such that $\left(\sum_{i=1}^{n-1}a_i^2\right)x^2+2\left(\sum_{i=1}^{n-1}a_i\,a_{i+1}\right)x+\sum_{i=2}^{n}a_i^2\leq 0$,

then $a_1, a_2, a_3, \ldots$ are in

(a) AP *(b)* GP

(c) HP *(d)* None of these

4. A function $f: Z\to Z$ is defined as $f(x)=\begin{cases}n+3, & n\in \text{odd}\\ \frac{n}{2}, & n\in \text{even}\end{cases}$ and let $p\in$ odd and $f[f\{f(p)\}]=27$, the sum of digits of p is

(a) 10 *(b)* 9 *(c)* 6 *(d)* 4

5. If z is a complex number and $\min(|z|+|z-2|+|2z-4|)=\lambda$ and if $y=2[x]+3=3[x-\lambda]$. Then, the value of $[x+y]$ is (where $[\cdot]$ denotes the greatest integer function)

(a) 30 *(b)* 40 *(c)* 35 *(d)* 50

One or More than One Option Correct Type

6. Let $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$ be three unit vectors such that $\hat{\mathbf{u}}+\hat{\mathbf{v}}+\hat{\mathbf{w}}=\hat{\mathbf{a}}$, $\hat{\mathbf{a}}\cdot\hat{\mathbf{u}}=\frac{3}{2}$, $\hat{\mathbf{a}}\cdot\hat{\mathbf{v}}=\frac{7}{4}$ and $|\hat{\mathbf{a}}|=2$, then

(a) $\hat{\mathbf{u}}\cdot\hat{\mathbf{v}}=\frac{3}{4}$ *(b)* $\hat{\mathbf{u}}\cdot\hat{\mathbf{w}}=0$ *(c)* $\hat{\mathbf{u}}\cdot\hat{\mathbf{v}}=\frac{1}{4}$ *(d)* $\hat{\mathbf{u}}\cdot\hat{\mathbf{w}}=-\frac{1}{4}$

7. If the function f such that $f(xy)=f(x)+f(y)$; $x, y>0$, $f'(1)=2$ and A is the area bounded by the curves $y=f(x)$, $x=2$ and X-axis, then

(a) $f(x)=2\log_e x$ *(b)* $A=4\log\left(\frac{2}{\sqrt{e}}\right)$

(c) $A=2(2\log_e 2-1)$ *(d)* $f(x)=\log_e x$

8. If a tangent drawn to the curve $y=f(x)$ at $P(x, y)$ cuts the X-axis and Y-axis at the M and N, respectively such that $NP:MP=3:1$ and $f(1)=1$, then

(a) equation of curve is $x\frac{dy}{dx}+3y=0$

(b) equation of curve is $x\frac{dy}{dx}-3y=0$

(c) normal at $(1, 1)$ is $x+3y=4$

(d) curve passes through $\left(2, \frac{1}{8}\right)$

9. If $(1+x+x^2)^n=a_0+a_1x+a_2x^2+\ldots+a_{2n}x^{2n}$, then

(a) $a_0+a_1+a_2+\ldots+a_n=0$

(b) $a_0+a_3+a_6+\ldots=3^{n-1}$

(c) $a_0^2-a_1^2+a_2^2-\ldots+a_{2n}^2=a_n$

(d) $a_0-a_1+a_2-a_3+\ldots+a_{2n}=1$

10. The range of values of 'p' such that angle θ between the pair of tangent drawn from $(p, 0)$ to the circle $x^2+y^2=1$ satisfies $\frac{\pi}{2}<\theta<\pi$, lies in

(a) $(-4, 1)$ *(b)* $(1, \sqrt{2})$ *(c)* $(2, 3)$ *(d)* $(-\sqrt{2}, -1)$

11. If $f(x)=\begin{cases}37-x, & 2<x\le 3\\ 3x^2+12x-1, & -1\le x\le 2\end{cases}$, then

(a) $f'(2)$ exists

(b) $f(x)$ is increasing on $[-1,2]$

(c) $f(x)$ is continuous on $[-1,3]$

(d) $f(x)$ has the maximum value at $x=2$

12. If $z_1=a+ib$ and $z_2=c+id$ are complex numbers such that $|z_1|=|z_2|=1$ and $\text{Re}(z_1\bar{z}_2)=0$, then the pair of complex numbers $z_3=a+ic$ and $z_4=b+id$ satisfy

(a) $|z_3|=1$ (b) $|z_3|=3$

(c) $|z_4|=1$ (d) $\text{Re}|z_3\,\bar{z}_4|=0$

13. The probabilities that a student passes in Physics, Chemistry and Mathematics are p, c and m respectively. Of these subjects, a student has 75% chance of passing in atleast one, 50% chance of passing in atleast two and 40% chance of passing in exactly two subjects, then which of the following relations are true?

(a) $p+c+m=\frac{27}{20}$ (b) $p+c+m=\frac{19}{20}$

(c) $pcm=\frac{1}{4}$ (d) $pcm=\frac{1}{10}$

Single Integer Type Questions

14. Let $A=\begin{bmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{bmatrix}$ and $10B=\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & p\\ 1 & -2 & 3\end{bmatrix}$. If B is the inverse of A, then the value of p is ..... .

15. If $x_1=\sqrt{3}$ and $x_{n+1}=\frac{x_n}{1+\sqrt{1+x_n^2}}$, $\forall\, n\in N$, then the value of $\frac{3}{\pi}\left(\lim\limits_{x\to\infty}2^n x_n\right)$ is equal to .... .

16. The line $3x+2y=24$ meets the Y-axis at A and the X-axis at B. If the perpendicular bisector of AB meets the line through $(0,-1)$ parallel to X-axis at P, then the value of (area of $\Delta ABP-90$) is .... .

17. If a,b,c are three distinct positive numbers, each different from 1 such that $(\log_b a\cdot\log_c a-\log_a a)+(\log_a b\cdot\log_c b-\log_b b)+(\log_a c\cdot\log_b c-\log_c c)=0$, then the value of abc is ..... .

18. The sum of the series $\left[\frac{8}{5}+\frac{16}{65}+\ldots+\frac{128}{2^{18}+1}\right]$ is .... (where $[\cdot]$ is the greatest integer value).

19. If $r=x_1+x_2+x_3$ and $\tan^{-1}\left(\sqrt{\frac{x_1 r}{x_2x_3}}\right)+\tan^{-1}\left(\sqrt{\frac{x_2 r}{x_1x_3}}\right)+\tan^{-1}\left(\sqrt{\frac{x_3 r}{x_1x_2}}\right)=k\pi$, then the value of k is .... .

20. If $\left|\frac{12x}{4x^2+9}\right|\le 1$, then the real value of $\left(|x|+\frac{1}{2}\right)$ which being satisfied the above inequality, is ..... .

21. The value of $\lim\limits_{n\to\infty}\left[\tan\frac{\pi}{2n}\tan\frac{2\pi}{2n}\ldots\tan\frac{n\pi}{2n}\right]^{\frac{1}{n}}$ is .... .

Linked Comprehension Type Questions

Passage (Q. Nos. 22-23) Consider, the function $f(x)=\max\{x^2,(1-x)^2,2x(1-x)\}$, $x\in[0,1]$.

22. The interval in which $f(x)$ is decreasing, is

(a) $\left(\frac{1}{3},\frac{1}{2}\right)\cup\left(\frac{1}{2},1\right)$ (b) $\left(0,\frac{1}{3}\right)\cup\left(\frac{1}{2},\frac{2}{3}\right)$

(c) $\left(\frac{1}{3},1\right)$ (d) $\left(\frac{1}{3},\frac{2}{3}\right)$

23. If Rolle's theorem is applicable for $f(x)$ on (a,b), then $a+b+c$ equals (where, c is the point such that $f'(c)=0$)

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) 1

Passage (Q. Nos. 24-25) A point P moves such that sum of the slopes of the normals drawn from it to the hyperbola $xy=16$ is equal to the sum of ordinates of feet of normal. The locus of P is a curve C.

24. The equation of the curve C is

(a) $x^2=16y$ (b) $x^2=25y$

(c) $x^2=10y$ (d) $x^2=9y$

25. If the tangent to the curve C cuts the coordinate axes at M and N, then the locus of the middle point of MN is

(a) $y^2=4x$ (b) $x^2+2y=0$

(c) $x^2+6y=0$ (d) $x^2+10y=0$

Passage (Q. Nos. 26-27) There are n students in a class and probability that exactly λ out of n pass the examination is directly proportional to λ^2 $(0\le\lambda\le n)$.

26. The probability that a student selected at random who passed the examination, is

(a) $\frac{(n+1)}{2(2n+1)}$ (b) $\frac{3(n+1)}{2(2n+1)}$

(c) $\frac{2(n+1)}{3(2n+1)}$ (d) $\frac{2(n+1)}{(2n+1)}$

27. If a selected student has been found to pass the examination, then the probability that he is the only student to have passed the examination, is

(a) $\frac{4}{[n(n+1)]^2}$ (b) $\frac{2}{[n(n+1]^2}$

(c) $\frac{1}{[n(n+1)^2]}$ (d) $\frac{5}{[n(n+1)]^2}$

Match the Columns

28. Match the statements of Column I with the values of Column II.

Column I	Column II
A. If the line $\frac{x-1}{1}=\frac{y+1}{-2}=\frac{z+1}{\lambda}$ lies in the plane $3x-2y+5z=0$, then λ is equal to	p. $\sin^{-1}\left(\sqrt{\frac{6}{25}}\right)$
B. If $(3, \lambda, \mu)$ is a point on the lines $2x+y+z-3=0=x-2y+z-1$, then $\lambda+\mu$ is equal to	q. $-\frac{7}{5}$
C. The angle between the lines $x=y=z$ and the plane $4x-3y+5z=2$ is	r. -3
D. The angle between the planes $x+y+z=0$ and $3x-4y+5z=0$ is	s. $\cos^{-1}\left(\sqrt{\frac{8}{75}}\right)$

29. Match the following columns.

Column I	Column II
A. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three mutually perpendicular vectors, where $\lvert\mathbf{a}\rvert=\lvert\mathbf{b}\rvert=3, \lvert\mathbf{c}\rvert=2$, then $[\mathbf{a}\times\mathbf{b}\ \mathbf{b}\times\mathbf{c}\ \mathbf{c}\times\mathbf{a}]$ is	p. $90°$
B. If $\lvert\mathbf{a}\ \ \mathbf{b}\rvert=\lvert\mathbf{a}-\mathbf{b}\rvert$, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is	q. 324
C. If $\lvert\mathbf{a}\rvert=\lvert\mathbf{b}\rvert=\lvert\mathbf{c}\rvert$, angle between each pair of vectors is $\frac{\pi}{3}$ and $\lvert\mathbf{a}+\mathbf{b}+\mathbf{c}\rvert=\sqrt{6}$, then $\lvert\mathbf{a}\rvert$ is equal to	r. $-6, -1$
D. If $\mathbf{a}=2\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}+3\hat{\mathbf{k}}$, $\mathbf{b}=2\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\mathbf{c}=3\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\mathbf{a}+\lambda\mathbf{b}$ is perpendicular to $\mathbf{c}$, then $\lvert\lambda\rvert$ is	s. 1

30. Match the following columns.

Column I	Column II
A. If $\Delta(x)=\begin{vmatrix} x^2-5x+3 & 2x-5 & 3 \\ 3x^2+x+4 & 6x+1 & 9 \\ 7x^2-6x+9 & 14x-6 & 21 \end{vmatrix}$ $=ax^3+bx^2+cx+d$	p. $c-d=119$
B. If $\Delta(x)=\begin{vmatrix} x-1 & 5x & 7 \\ x^2-1 & x-1 & 8 \\ 2x & 3x & 0 \end{vmatrix}$ $=ax^3+bx^2+cx+d$	q. $a+2b+3c+5d=156$
C. If $\Delta(x)=\begin{vmatrix} 2x^3-3x^2 & 5x+7 & 2 \\ 4x^3-7x & 3x+2 & 1 \\ 7x^3-8x^2 & x-1 & 3 \end{vmatrix}$ $=a+bx+cx^2+dx^3+ex^4$	r. $3a+4b+5c+d=141$
	s. $b-c=25$
	t. $3a+2b+5c+5d=187$

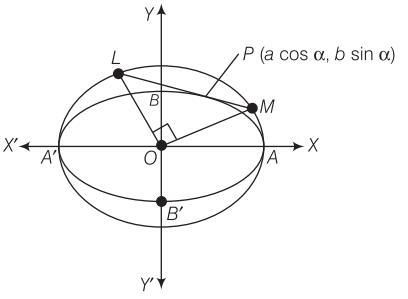
Answers with Explanation

1. ***(b)*** Given equation of an ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \ldots(i)$$

and equation of its auxiliary circle is

$$x^2+y^2=a^2 \quad \ldots(ii)$$



Let $P(a\cos\alpha, b\sin\alpha)$ be the point on the ellipse, then the equation of tangent to the point P is

$$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1 \quad \ldots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$x^2+y^2-a^2\left(\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}\right)^2=0$$

$$\Rightarrow (1-\cos^2\alpha)^2x^2+\left(1-\frac{a^2}{b^2}\sin^2\alpha\right)y^2 - 2\left(\frac{a}{b}\right)\cos\alpha\sin\alpha\ x\cdot y=0 \quad \ldots(iv)$$

Since, $OL \perp OM$

$\therefore$ Coefficient of x^2 + Coefficient of $y^2=0$

$$\Rightarrow 1-\cos^2\alpha+1-\frac{a^2}{b^2}\sin^2\alpha=0$$

$$\Rightarrow \sin^2\alpha\left(\frac{a^2}{b^2}-1\right)=1$$

$$\Rightarrow \sin^2\alpha\left(\frac{1}{1-e^2}-1\right)=1$$

$$\Rightarrow e^2\sin^2\alpha=1-e^2 \Rightarrow e^2(1+\sin^2\alpha)=1$$

$$\therefore e=\frac{1}{\sqrt{1+\sin^2\alpha}}$$

2. ***(c)*** Let $g(x)=ax^2+bx+c>0, \forall\ x\in R$

$$\Rightarrow b^2-4ac<0 \text{ and } a>0 \quad \ldots(i)$$

Now, $h(x)=g(x)-g'(x)+g''(x)$

$$=(ax^2+bx+c)-(2ax+b)+2a$$

$$=ax^2+(b-2a)x+(2a-b+c)$$

$$\Rightarrow \Delta=(b-2a)^2-4a(2a-b+c)$$

$$=(b^2-4ac)-4a^2<0 \quad \text{[using Eq. (i)]}$$

$\therefore\ h(x)>0, \forall x\in R$

3. ***(b)*** It is given that

$$\left(\sum_{i=1}^{n-1}a_i^2\right)x^2+2\left(\sum_{i=1}^{n-1}a_ia_{i+1}\right)x+\sum_{i=2}^{n}a_i^2\le 0$$

$$\Rightarrow \quad \sum_{i=1}^{n-1}(a_i x + a_{i+1})^2 \le 0$$

$$\Rightarrow \quad \sum_{i=1}^{n-1}(a_i x + a_{i+1})^2 = 0$$

$$\Rightarrow \quad a_i x + a_{i+1} = 0,\ \forall i = 1,2,3,\ldots,n-1$$

$$\Rightarrow \quad \frac{a_{i+1}}{a_i} = -x,\ \forall i = 1,2,3,\ldots,n-1$$

$$\Rightarrow \quad \frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots = \frac{a_n}{a_{n+1}}$$

$\therefore$ $a_1, a_2, a_3, \ldots, a_n$ are in GP.

4. ***(c)*** Since, $p \in$ odd

$$\therefore \quad f(x) = p + 3 \ \Rightarrow \ f\{f(p)\} = \frac{p+3}{2}$$

If $\dfrac{p+3}{2} = \in$ odd $\Rightarrow \quad 27 = \dfrac{p+3}{2} + 3$

$\therefore$ $p = 45$ is not possible

and now let $\dfrac{p+3}{2} \in$ even

$$\therefore \quad 27 = f[f\{f(p)\}] = f\left(\frac{p+3}{2}\right) = \frac{p+3}{4} \Rightarrow p = 105$$

Verifying $f[f\{f(105)\}] = f\{f(100)\}$

$= f(54) = 27$

Thus, sum of digits of p is $1 + 0 + 5 = 6$

5. ***(a)*** We have,

$$|z| + |z-2| + |2z-4| = |z| + |z-2| + |4-2z|$$
$$\ge |z + z - 2 + 4 - 2z| = 2$$

$$\Rightarrow \quad |z| + |z-2| + |2z-4| \ge 2$$

$$\therefore \quad \lambda = 2$$

Then, $2[x] + 3 = 3[x - \lambda] = 3[x-2]$

$$\Rightarrow \quad 2[x] + 3 = 3([x] - 2)$$

$$\Rightarrow \quad [x] = 9$$

Then, $y = 2 \cdot 9 + 3 = 18 + 3 = 21$

$$\therefore \quad [x + y] = [9 + 21] = 30$$

6. ***(a, d)*** We have, $\hat{\mathbf{u}} + \hat{\mathbf{v}} + \hat{\mathbf{w}} = \hat{\mathbf{a}}$ ...(i)

Taking dot product on both sides of Eq. (i) with $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$ and $\hat{\mathbf{a}}$ respectively, we get

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} + \hat{\mathbf{u}} \cdot \hat{\mathbf{w}} = \frac{1}{2} \quad \ldots(ii)$$

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} + \hat{\mathbf{v}} \cdot \hat{\mathbf{w}} = \frac{3}{4} \quad \ldots(iii)$$

$$\hat{\mathbf{a}} \cdot \hat{\mathbf{w}} = \frac{3}{4} \quad \ldots(iv)$$

Again, taking dot product on both sides of Eq. (i) with $\hat{\mathbf{w}}$, we get

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{w}} + \hat{\mathbf{v}} \cdot \hat{\mathbf{w}} + |\hat{\mathbf{w}}|^2 = \hat{\mathbf{a}} \cdot \hat{\mathbf{w}}$$

$$\Rightarrow \quad \hat{\mathbf{u}} \cdot \hat{\mathbf{w}} + \hat{\mathbf{u}} \cdot \hat{\mathbf{w}} + 1 = \frac{3}{4}$$

$$\Rightarrow \quad \hat{\mathbf{u}} \cdot \hat{\mathbf{w}} + \hat{\mathbf{v}} \cdot \hat{\mathbf{w}} = -\frac{1}{4} \quad \ldots(v)$$

On adding Eqs. (ii), (iii) and (v), we get

$$(\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} + \hat{\mathbf{v}} \cdot \hat{\mathbf{w}} + \hat{\mathbf{u}} \cdot \hat{\mathbf{w}}) = \frac{1}{2} \quad \ldots(vi)$$

Now, on subtracting Eq. (ii) from Eq. (vi), we get

$$\hat{\mathbf{v}} \cdot \hat{\mathbf{w}} = 0$$

On subtracting Eq. (iii) from Eq. (vi), we get

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{w}} = -\frac{1}{4}$$

From Eq. (vi), $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = \dfrac{3}{4}$

7. ***(a, b, c)*** We have,

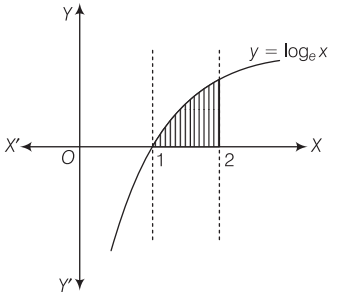
$$f(xy) = f(x) + f(y) \quad \ldots(i)$$

Now, $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f\left(x\left(1 + \frac{h}{x}\right)\right) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x) + f\left(1 + \frac{h}{x}\right) - f(x)}{h} \quad \text{[using Eq. (i)]}$$

$$= \lim_{h \to 0} \frac{f\left(1 + \frac{h}{x}\right)}{h} \left[\because \frac{0}{0} \text{form and from Eq. (i), } f(1) = 0\right]$$



$$\Rightarrow \quad f'(x) = \frac{1}{x} f'(1)$$

$$\Rightarrow \quad f'(x) = \frac{2}{x} \quad [\because f'(1) = 2]$$

$$\Rightarrow \quad f(x) = 2\log_e x + C \quad \ldots(ii)$$

$$\Rightarrow \quad f(x) = 2\log_e x \quad [\because f(1) = 0, c = 0]$$

$$\therefore \quad \text{Area} = 2\int_1^2 \log_e x \, dx$$

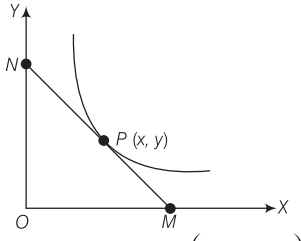
$$= 2[x \log_e x - x]_1^2$$

$$= 2(2\log_e 2 - 1) = 4\log\left(\frac{2}{\sqrt{e}}\right)$$

8. ***(a, d)*** Equation of tangent to the curve $y = f(x)$ at $P(x, y)$ is

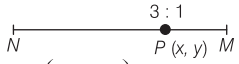
$$Y - y = \frac{dy}{dx}(X - x)$$

$$\Rightarrow \quad Y - y = y_1(X - x) \quad \ldots(i)$$



This tangent cuts X-axis at $M\left(x - \dfrac{y}{y_1}, 0\right)$ and Y-axis at $N(0, y - xy_1)$.

Since, $P(x, y)$ divides NM externally in the ratio $3:1$.



$$\therefore \quad x = \frac{3\left(x - \frac{y}{y_1}\right) + 1 \cdot 0}{4}$$

$$\Rightarrow \quad 4x = 3x - \frac{3y}{y_1} \Rightarrow xy_1 = -3y$$

$$\Rightarrow \quad xy_1 + 3y = 0 \qquad \ldots(ii)$$

and $$y = \frac{3 \times 0 + 1(y - xy_1)}{4}$$

$$\Rightarrow \quad 3y + xy_1 = 0 \qquad \ldots(iii)$$

From Eqs. (ii) and (iii), we get

$$xy_1 + 3y = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-3y}{x} \qquad \ldots(iv)$$

$$\Rightarrow \quad \frac{dy}{y} = \frac{-3}{x}$$

$$\Rightarrow \quad \int \frac{dy}{y} = -3\int \frac{dx}{x}$$

$$\Rightarrow \quad \log y = -3\log x + \log C$$

$$\Rightarrow \quad \log (yx^3) = \log C$$

$$\Rightarrow \quad yx^3 = C \qquad \ldots(v)$$

When $x = 1$, $y = 1$, then $c = 1$

From Eq. (v), equation of curve, $y = \frac{1}{x^3}$ $\quad [\because f(1) = 1]$

$\therefore$ This curve passing through $\left(2, \frac{1}{8}\right)$.

9. ***(b, c, d)*** We have,

$$(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n} \qquad \ldots(i)$$

At $x = 1$, from Eq. (i)

$$3^n = a_0 + a_1 + a_2 + \ldots + a_{2n} \qquad \ldots(ii)$$

At $x = -1$, from Eq. (i)

$$1^n = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 + \ldots + (-1)^{2n}a_{2n}$$

$$\Rightarrow \quad a_0 - a_1 + a_2 - a_3 + \ldots + a_{2n} = 1 \qquad \ldots(iii)$$

At $x = w$, from Eq. (i)

$$0 = a_0 + a_1 + a_2 + \ldots + a_{2n} \qquad \ldots(iv)$$

At $x = w^2$, from Eq. (i)

$$0 = a_0 + a_1w^2 + a_2w + a_3 + a_4w^2 + \ldots + a_{2n}w^{2n} \qquad \ldots(v)$$

On adding Eqs. (ii), (iv) and (v), we get

$$3^n = 3(a_0 + a_3 + a_6 + \ldots)$$

$$\Rightarrow \quad a_0 + a_3 + a_6 + \ldots = \frac{3^n}{3} = 3^{n-1}$$

Replacing x by $-\frac{1}{x}$ in Eq. (i), we get

$$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \ldots + \frac{a_{2n}}{x^{2n}}$$

$$\Rightarrow (1 + x^2 - x)^n = a_0x^{2n} - a_1x^{2n-1} + a_2x^{2n-2} - \ldots + a_{2n} \qquad \ldots(vi)$$

On multiplying Eqs. (i) and (vi), we get

$$(a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}) \times (a_0x^{2n} - a_1x^{2n-1} + a_2x^{2n-2} - \ldots + a_{2n})$$
$$= [(1 + x^2)^2 - x^2]^n = (1 + x^2 + x^4)^n \qquad \ldots(vii)$$

Replacing x by x^2 in Eq. (i), we get

$$(1 + x^2 + x^4)^n = a_0 + a_1x^2 + a_2x^4 + \ldots + a_nx^{2n} + \ldots + a_{2n}x^{4x}$$

Here, coefficient of x^{2n} is a_n.

From Eq. (vii), equating the coefficient of x^{2n}, we get

$$a_0^2 - a_1^2 + \ldots + a_{2n}^2 = a_n$$

10. ***(b, d)*** We have, equation of circle

$$x^2 + y^2 = 1$$

$\therefore$ Equation of pair of tangents is

$$(px + 0 - 1)^2 = (x^2 + y^2 - 1)(p^2 + 0 - 1)$$

$$\Rightarrow \quad (p^2 - 1)y^2 - x^2 + 2px - p^2 = 0$$

If θ is the angle between the tangents, then

$$\tan\theta = \frac{2\sqrt{-(p^2 - 1)(-1)}}{p^2 - 2} = \frac{2\sqrt{p^2 - 1}}{p^2 - 2}$$

Since, θ lies between $\frac{\pi}{2} < \theta < \pi$, then $\tan\theta < 0$

$$\therefore \quad \frac{2\sqrt{p^2 - 1}}{p^2 - 2} < 0$$

$$\Rightarrow \quad p^2 - 1 > 0 \text{ and } p^2 - 2 < 0$$

$$\Rightarrow \quad |p| > 1 \text{ and } |p| < \sqrt{2}$$

$$\Rightarrow \quad p \in (-\sqrt{2}, -1) \cup (1, \sqrt{2})$$

11. ***(b, c, d)*** Given that, $f(x) = 3x^2 + 12x - 1, -1 \le x \le 2$

$$\Rightarrow \quad f'(x) = 6x + 12$$

$f(x)$ is increasing.

$$\Rightarrow \quad f'(x) \ge 0$$

$$\Rightarrow \quad 6x + 12 \ge 0 \Rightarrow x \ge -2$$

$\Rightarrow f(x)$ is increasing in $[-1, 2]$.

Now, $$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2 - h)$$
$$= \lim_{h \to 0} \{3(2 - h)^2 + 12(2 - h) - 1\} = 35$$

and $$\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2 + h)$$
$$= \lim_{h \to 0} \{37 - (2 + h)\} = 35$$

Hence, $f(x)$ is continuous on $[-1, 3]$.

$$f'(2^-) = 24 \text{ and } f'(2^+) = -1$$

$\Rightarrow f'(2)$ does not exist.

For maximum, $f(2) = 35$

$$f(-1) = -10$$
$$f(3) = 34$$

Hence, $f(x)$ has maximum value at $x = 2$.

12. ***(a, c, d)*** We have, $z_1 = a + ib$

and $z_2 = c + id$

$$\left.\begin{aligned} \therefore \quad |z_1|^2 &= a^2 + b^2 = 1 \\ \text{and} \quad |z_2|^2 &= c^2 + d^2 = 1 \end{aligned}\right\} \qquad \ldots(i)$$

Also, $\mathrm{Re}(z_1\bar{z}_2) = (ac + bd) = 0$

$$\Rightarrow \quad \frac{a}{b} = \frac{-d}{c} = \lambda \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$b^2\lambda^2 + b^2 = c^2 + \lambda^2c^2$$

$$\Rightarrow \quad b^2 = c^2 \text{ and } a^2 = d^2$$

Now, $|z_3| = \sqrt{a^2 + c^2} = \sqrt{a^2 + b^2} = 1$

$|z_4| = \sqrt{b^2 + d^2} = \sqrt{a^2 + b^2} = 1$

$\therefore \quad \text{Re}|z_3 \bar{z}_4| = (ab + cd)$

$= (b\lambda)b + c(-\lambda c) = \lambda(b^2 - c^2) = 0$

13. ***(a, d)*** We have,

P (Physics) $= p$

P (Chemistry) $= c$

P (Mathematics) $= m$

The probability of passing in atleast one subject,

$$\frac{75}{100} = 1 - (1-p)(1-c)(1-m)$$

$$\Rightarrow \quad \frac{3}{4} = p + c + m - mp - pc - mpc \qquad \ldots(i)$$

The probability of passing in atleast two subjects,

$$\frac{1}{2} = pcm + p(1-c)m + (1-p)mc + pc(1-m)$$

$$\Rightarrow \quad 2pcm = pm + cm + pc - \frac{1}{2} \qquad \ldots(ii)$$

The probability of passing in exactly two subjects,

$$\frac{2}{5} = (1-p)\,cm + p(1-c)m + pc\,(1-m)$$

$$\Rightarrow \quad \frac{2}{5} = pm + cm + pc - 3pcm \qquad \ldots(iii)$$

From Eqs. (ii) and (iii), we get

$$2pcm + \frac{1}{2} = \frac{2}{5} + 3pcm$$

$$\Rightarrow \quad pcm = \frac{1}{2} - \frac{2}{5} = \frac{1}{10}$$

From Eq. (ii), we get

$$\frac{1}{5} = pm + cm + pc - \frac{1}{2}$$

$$\Rightarrow \quad pm + cm + cp = \frac{1}{5} + \frac{1}{2} = \frac{7}{10}$$

From Eq. (i), we get

$$\frac{3}{4} = p + c + m - \frac{7}{10} + \frac{1}{10}$$

$$\Rightarrow \quad p + c + m = \frac{27}{20}$$

14. ***(5)*** $|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$

$= 1(1+3) + 1(2+3) + 1(2-1)$

$= 4 + 5 + 1 = 10$

$$\text{adj}(A) = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\Rightarrow \quad B = A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\Rightarrow \quad 10(B) = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$\therefore \quad p = 5$

15. ***(2)*** Let $x_n = \tan\theta_n$

Now, $\tan\theta_{n+1} = x_{n+1} = \dfrac{x_n}{1 + \sqrt{1 + x_n^2}}$

$$= \frac{\tan\theta_n}{1 + \sqrt{1 + \tan^2\theta_n}}$$

$$\Rightarrow \quad \tan\theta_{n+1} = \frac{\tan\theta_n}{1 + \sec\theta_n}$$

$$\Rightarrow \quad \frac{\sin\theta_n}{1 + \cos\theta_n} = \tan\frac{\theta_n}{2}$$

$$\Rightarrow \quad \theta_{n+1} = \frac{\theta_n}{2}$$

$$\Rightarrow \quad \theta_n = \frac{\theta_{n-1}}{2}$$

Now, $\theta_1 = \dfrac{\pi}{3}$

$$\Rightarrow \quad \theta_n = \frac{\pi}{3 \cdot 2^{n-1}}$$

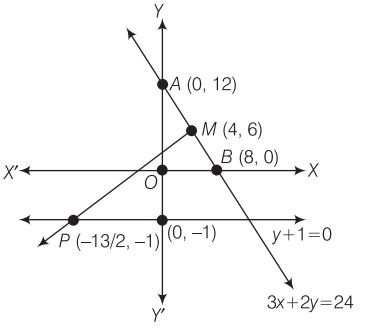
$$\Rightarrow \quad x_n = \tan\left(\frac{2\pi}{3 \cdot 2^n}\right)$$

$$\Rightarrow \quad \lim_{x \to \infty} 2^n \cdot x_n = \lim_{n \to \infty} \frac{\tan\left(\dfrac{2\pi}{3 \cdot 2^n}\right)}{\left(\dfrac{1}{2^n}\right)} = \frac{2\pi}{3}$$

$\therefore \quad \dfrac{3}{\pi} \lim\limits_{x \to \infty} 2^n x_n = 2$

16. ***(1)*** The given line $3x + 2y = 24$ meet Y-axis at $A(0,12)$ and X-axis at $B(8,0)$. The mid-point of AB is $M(4,6)$. The equation of perpendicular bisector of AB is

$$2x - 3y = k \qquad \ldots(i)$$



Since, line (i) passes through (4, 6).

$\therefore \quad k = -10$

$\therefore$ The equation of the right bisector of AB is

$$2x - 3y + 10 = 0 \qquad \ldots(ii)$$

The line through $(0, -1)$ parallel to OX is

$$y + 1 = 0 \qquad \ldots(iii)$$

Lines (ii) and (iii) meet at $P\left(\dfrac{-13}{2}, -1\right)$.

$\therefore$ Area of $\Delta ABP = \dfrac{1}{2} \times AB \times PM$

$$= \frac{1}{2} \times (4\sqrt{13}) \times \left(\frac{7}{2}\sqrt{13}\right)$$

$= 91$ sq units

$\therefore$ (Area of $\Delta ABP - 90$) $= 1$ sq unit

17. *(1)* We have,

$$(\log_b a \cdot \log_c a - \log_a a) + (\log_a b \cdot \log_c b - \log_b b) + (\log_a c \cdot \log_b c - \log_c c) = 0$$

$$\Rightarrow \left(\frac{\log a}{\log b}\cdot\frac{\log a}{\log c} - 1\right) + \left(\frac{\log b}{\log a}\cdot\frac{\log b}{\log c} - 1\right) + \left(\frac{\log c}{\log a}\cdot\frac{\log c}{\log b} - 1\right) = 0$$

Let $\log a = x$, $\log b = y$ and $\log c = z$, then

$$\left(\frac{x^2}{yz} - 1\right) + \left(\frac{y^2}{xz} - 1\right) + \left(\frac{z^2}{xy} - 1\right) = 0$$

$$\Rightarrow \frac{x^2}{yz} + \frac{y^2}{xz} + \frac{z^2}{xy} - 3 = 0$$

$$\Rightarrow \frac{x^3 + y^3 + z^3 - 3xyz}{xyz} = 0$$

$$\Rightarrow x^3 + y^3 + z^3 - 3xyz = 0$$

$$\Rightarrow (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = 0$$

$$\Rightarrow (x + y + z)\cdot\frac{1}{2}[(x - y)^2 + (y - z)^2 + (z - x)^2] = 0$$

$$\Rightarrow (x + y + z)[(x - y)^2 + (y - z)^2 + (z - x)^2] = 0 \quad \ldots(i)$$

Since, $a \neq b \neq c$

$\Rightarrow \log a \neq \log b \neq \log c$

$\Rightarrow x \neq y \neq z$

$\Rightarrow$ second factor of Eq. (i) $\neq 0$

$\Rightarrow$ first factor of Eq. (i) $= 0$

$\Rightarrow x + y + z = 0$

$\Rightarrow \log a + \log b + \log c = 0$

$\Rightarrow \log(abc) = 0$

$\Rightarrow abc = 1$

18. *(1)* We have, $\frac{8}{5} + \frac{16}{65} + \ldots + \frac{128}{2^{18} + 1}$

$$S_n = \sum_{r=1}^{n} \frac{8r}{4r^4 + 1}$$

$$= \sum_{r=1}^{n} \frac{8r}{(2r^2 - 2r + 1)(2r^2 + 2r + 1)}$$

$$= 2\sum_{r=1}^{n} \frac{(2r^2 + 2r + 1) - (2r^2 - 2r + 1)}{(2r^2 - 2r + 1)(2r^2 + 2r + 1)}$$

$$= 2\sum_{r=1}^{n} \left(\frac{1}{2r^2 - 2r + 1} - \frac{1}{2r^2 + 2r + 1}\right)$$

$$= 2\left\{\frac{1}{1} - \frac{1}{5} + \frac{1}{5} - \frac{1}{13} + \ldots + \frac{1}{2n^2 - 2n + 1} - \frac{1}{2n^2 + 2n + 1}\right\}$$

$$= 2\left\{1 - \frac{1}{2n^2 + 2n + 1}\right\}$$

$$= \frac{2(2n^2 + 2n)}{2n^2 + 2n + 1} = \frac{4n^2 + 4n}{2n^2 + 2n + 1}$$

$$\Rightarrow S_{16} = \frac{1088}{545}$$

$\therefore [S_{16}] = 1$

19. *(1)* Let $\tan^{-1}\left(\sqrt{\frac{x_1 r}{x_2 x_3}}\right) = \alpha$, $\tan^{-1}\left(\sqrt{\frac{x_2 r}{x_1 x_3}}\right) = \beta$ and $\tan^{-1}\left(\sqrt{\frac{x_3 r}{x_1 x_2}}\right) = \gamma$

$$\therefore \tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = \tan^{-1}\left[\frac{\alpha + \beta + \gamma - 2\beta\gamma}{1 - \alpha\beta - \beta\gamma - \gamma\alpha}\right]$$

Then, $\alpha + \beta + \gamma - \alpha\beta\gamma = \sqrt{\frac{x_1 r}{x_2 x_3}} + \sqrt{\frac{x_2 r}{x_1 x_3}} + \sqrt{\frac{x_3 r}{x_1 x_2}} - \sqrt{\frac{x_1 r}{x_2 x_3}}\sqrt{\frac{x_2 r}{x_1 x_3}}\sqrt{\frac{x_3 r}{x_1 x_2}}$

$$= \frac{x_1\sqrt{r} + x_2\sqrt{r} + x_3\sqrt{r}}{\sqrt{x_1 x_2 x_3}} - \frac{r\sqrt{r}}{\sqrt{x_1 x_2 x_3}}$$

$$= \frac{r\sqrt{r}}{\sqrt{x_1 x_2 x_3}} - \frac{r\sqrt{r}}{\sqrt{x_1 x_2 x_3}} = 0$$

and $1 - \alpha\beta - \beta\gamma - \gamma\alpha$

$$= 1 - \sqrt{\frac{x_1 r}{x_2 x_3}}\cdot\sqrt{\frac{x_2 r}{x_1 x_3}} - \sqrt{\frac{x_2 r}{x_1 x_3}}\cdot\sqrt{\frac{x_3 r}{x_1 x_2}} - \sqrt{\frac{x_3 r}{x_1 x_2}}\cdot\sqrt{\frac{x_1 r}{x_2 x_3}}$$

$$= 1 - \frac{r}{x_3} - \frac{r}{x_1} - \frac{r}{x_2}$$

$$= 1 - r\left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right] \neq 0$$

$\therefore$ LHS $= \tan^{-1} 0 = n\pi = -\pi, 0, \pi$ [for principal values]

$= \pi$ [sum of three positive angles cannot be zero or negative]

$\therefore \quad k = 1$

20. *(2)* We have, $-1 \leq \frac{12x}{4x^2 + 9} \leq 1$

Let $y = \frac{12x}{4x^2 + 9}$

$$\Rightarrow 4yx^2 - 12x + 9y = 0$$

For x is real, $144 - 4\cdot 4y \cdot 9y \geq 0$

$$\Rightarrow 144 - 144y^2 \geq 0$$

$$\Rightarrow 1 - y^2 \geq 0 \Rightarrow y^2 \leq 1$$

$$\Rightarrow -1 \leq y \leq 1 \Rightarrow |y| \leq 1$$

$$\Rightarrow \left|\frac{12x}{4x^2 + 9}\right| = 1 \Rightarrow \frac{|12x|}{|4x^2 + 9|} = 1$$

$$\Rightarrow \frac{12|x|}{|4x^2 + 9|} = 1$$

$$\Rightarrow 12|x| = |4x^2| + 9$$

$$\Rightarrow |4x^2| - 12|x| + 9 = 0$$

$$\Rightarrow (2|x| - 3)^2 = 0$$

$$\Rightarrow 2|x| = 3$$

$$\Rightarrow |x| = \frac{3}{2}$$

$$\therefore \left(|x| + \frac{1}{2}\right) = \frac{3}{2} + \frac{1}{2}$$

$$= 2$$

21. *(1)* Let $P = \lim_{n \to \infty} \left[\tan\frac{\pi}{2n} \tan\frac{2\pi}{2n} \ldots \tan\frac{n\pi}{2n}\right]^{\frac{1}{n}}$

Taking logarithm on both sides, we get

$$\log P = \lim_{n \to \infty} \frac{1}{n}\left[\log \tan\left(\frac{\pi}{2n}\right) + \log \tan\left(\frac{2\pi}{2n}\right) + \ldots + \log \tan\left(\frac{n\pi}{2n}\right)\right]$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \log \tan\left(\frac{r\pi}{2n}\right)$$

$$\log P = \int_0^1 \log \tan\left(\frac{x\pi}{2}\right) dx \quad \ldots(i)$$

$$= \int_0^1 \log \tan\frac{(1-x)\pi}{2}\, dx$$

$$= \int_0^1 \log \cot\left(\frac{x\pi}{2}\right) dx \quad \ldots(ii)$$

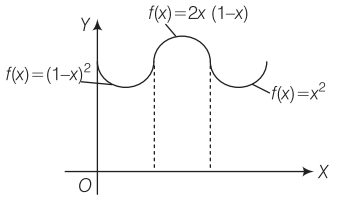
On adding Eqs. (i) and (ii), we get

$$2\log P = \int_0^1 \left[\log \tan\left(\frac{x\pi}{2}\right) + \log \cot\left(\frac{\pi x}{2}\right)\right] dx$$

$$= \int_0^1 \left[\log \tan\left(\frac{x\pi}{2}\right) . \cot\left(\frac{\pi x}{2}\right)\right] dx = \int_0^1 \log 1 = 0$$

$\Rightarrow \quad \log P = 0 \Rightarrow P = e^0 = 1$

Sol. (Q. Nos. 22-23) We draw the graphs of $f_1(x) = x^2$, $f_2(x) = (1-x)^2$ and $f_3(x) = 2x\,(1-x)$



$f(x)$ is defined as

$$f(x) = \begin{cases} (1-x)^2, & 0 \le x < \frac{1}{3} \\ 2x\,(1-x), & \frac{1}{3} \le x \le \frac{2}{3} \\ x^2, & \frac{2}{3} \le x \le 1 \end{cases}$$

Interval in decrease of $f(x)$ is $\left(0, \frac{1}{3}\right) \cup \left(\frac{1}{2}, \frac{2}{3}\right)$.

Clearly, Rolle's theorem is applicable on $\left[\frac{1}{3}, \frac{2}{3}\right]$, where $f(x) = 2x\,(1-x)$

$\Rightarrow \quad f'(c) = 2 - 4c = 0 \Rightarrow c = \frac{1}{2}$

$\Rightarrow \quad a + b + c = \frac{1}{3} + \frac{2}{3} + \frac{1}{2} = \frac{3}{2}$

22. ***(b)*** **23.** ***(c)***

24. ***(a)*** Given, equation of hyperbola $xy = 16$

So, any point on the hyperbola is $\left(4t, \frac{4}{t}\right)$.

Equation of the normal at this point is

$$y - \frac{4}{t} = t^2\,(x - 4t)$$

If the normal passes through $P(h,k)$.

Then, $\quad k - \frac{4}{t} = t^2\,(h - 4t)$

$\Rightarrow \quad 4t^4 - t^3 h + tk - 4 = 0$

This equation has roots t_1, t_2, t_3 and t_4 which are parameters of the four feet of normal on the hyperbola. Therefore,

$$\Sigma t_1 = \frac{h}{4}$$

$$\Sigma t_1 t_2 = 0$$

$$\Sigma\, t_1 t_2 t_3 = -\frac{k}{4}$$

and $\quad t_1 t_2 t_3 t_4 = -1$

$\therefore \quad \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = \frac{k}{4}$

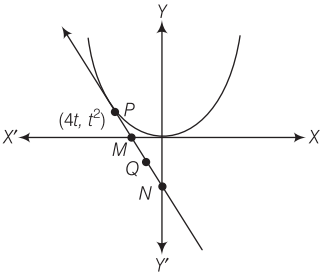
$\Rightarrow \quad y_1 + y_2 + y_3 + y_4 = k$

According to question,

$$t_1^2 + t_2^2 + t_3^2 + t_4^2 = \frac{h^2}{16} = k$$

Hence, the locus of $P(h,k)$ is $x^2 = 16y$.

25. ***(b)***



Equation of tangent at P is

$$x(4t) = \frac{16(y + t^2)}{2}$$

$\Rightarrow \quad 4tx = 8y + 8t^2$

$\Rightarrow \quad tx = 2y + 2t^2$

$\therefore \quad M(2t, 0), (0, -t^2)$

If $Q(h,k)$ is the middle point of MN.

Then, $\quad h = t, k = -\frac{t^2}{2}$

$\Rightarrow \quad 2k = -h^2$

Therefore, the locus of $Q(h,k)$ is $x^2 + 2y = 0$.

Sol. (Q. Nos. 26-27) Let E_λ be the event that exactly λ out of n pass the examination and A is the event that a student pass the examination, then $P(E_\lambda) \propto \lambda^2$

$\Rightarrow \quad P(E_\lambda) = k\lambda^2$ [k is proportional to constant]

$\because E_0, E_1, E_2, \ldots, E_n$ are mutually exclusive and exhaustive events.

$\Rightarrow \quad P(E_0) + P(E_1) + P(E_2) + \ldots + P(E_n) = 1$

$\Rightarrow \quad 0 + k(1)^2 + k(2)^2 + \ldots + k(n)^2 = 1$

$\Rightarrow \quad k(1^2 + 2^2 + 3^2 + \ldots + n^2) = 1$

$\Rightarrow \quad k \cdot \frac{n(n+1)(2n+1)}{6} = 1$

$$\Rightarrow \qquad k = \frac{6}{n(n+1)(2n+1)}$$

$$\therefore \quad P(A) = \sum_{\lambda=0}^{n} P(E_\lambda)\, P\left(\frac{A}{E_\lambda}\right)$$

$$= \sum_{\lambda=1}^{n} k\lambda^2 \times \frac{\lambda}{n} = \frac{k}{n}\sum_{\lambda=1}^{n} \lambda^3$$

$$= \frac{k}{n}\left(\frac{n(n+1)}{2}\right)^2 = \frac{3(n+1)}{2(2n+1)}$$

$$\text{and } P\left(\frac{E_1}{A}\right) = \frac{P(E_1)\, P\left(\frac{A}{E_1}\right)}{P(A)}$$

$$= \frac{\frac{6}{n(n+1)(2n+1)} \times \frac{1}{n}}{\frac{3(n+1)}{2(2n+1)}} = \frac{4}{[n(n+1)]^2}$$

26. ***(b)*** **27.** ***(a)***

28. A → q; B → r; C → p; D → s

(A) $3 \cdot 1 - 2(-2) + 5(\lambda) = 0$

$$\Rightarrow \qquad \lambda = \frac{-7}{5}$$

(B) Point $(3, \lambda, \mu)$ lies on

$$2x + y + z - 3 = x - 2y + z - 1$$

$$\Rightarrow \quad 2 \cdot 3 + \lambda + \mu - 3 = 0$$

and $\quad 3 - 2\lambda + \mu - 1 = 0$

$\Rightarrow 2\lambda - \mu - 2 = 0$ and $\lambda + \mu + 3 = 0$

So, $\quad \lambda + \mu = -3$

(C) $\sin\theta = \dfrac{1 \cdot 4 + 1 \cdot (-3) + 1 \cdot 5}{\sqrt{1^2 + 1^2 + 1^2}\sqrt{16 + 9 + 25}} = \dfrac{6}{\sqrt{3}\sqrt{50}}$

$$\therefore \quad \theta = \sin^{-1}\left(\sqrt{\frac{6}{25}}\right)$$

(D) $\cos\theta = \dfrac{1 \cdot 3 + 1 \cdot (-4) + 1 \cdot 5}{\sqrt{3}\sqrt{16 + 9 + 25}} = \dfrac{4}{\sqrt{3}\sqrt{50}}$

$$\theta = \cos^{-1}\left(\sqrt{\frac{8}{75}}\right)$$

29. A → q; B → p; C → s; D → r

(A) $[\mathbf{a} \times \mathbf{b}\ \mathbf{b} \times \mathbf{c}\ \mathbf{c} \times \mathbf{a}] = [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]^2$

$= (|\mathbf{a}||\mathbf{b}||\mathbf{c}|)^2 = 324$

(B) $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$

$$\Rightarrow \qquad |\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a} - \mathbf{b}|^2$$

$$\Rightarrow \qquad |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b}$$

$$\Rightarrow \qquad 4\mathbf{a}\cdot\mathbf{b} = 0 \ \Rightarrow \ \mathbf{a}\cdot\mathbf{b} = 0 \ \Rightarrow \ \mathbf{a} \perp \mathbf{b}$$

(C) $|\mathbf{a} + \mathbf{b} + \mathbf{c}| = \sqrt{6}$

$\Rightarrow \mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + 2\mathbf{a}\cdot\mathbf{b} + 2\mathbf{b}\cdot\mathbf{c} + 2\mathbf{c}\cdot\mathbf{a} = 6$

$\Rightarrow 6|\mathbf{a}|^2 = 6 \Rightarrow |\mathbf{a}| = 1$

(D) $\mathbf{a} + \lambda\mathbf{b} = 2(1+\lambda)\hat{\mathbf{i}} + (\lambda + \lambda^2)\hat{\mathbf{j}} + (3+\lambda)\hat{\mathbf{k}}$

Since, $(\mathbf{a} + \lambda\mathbf{b}) \perp \mathbf{c} \ \Rightarrow (\mathbf{a} + \lambda\mathbf{b})\cdot\mathbf{c} = 0$

$$\Rightarrow \quad 6(1+\lambda) + (\lambda + \lambda^2) + 0 = 0$$

$$\Rightarrow \qquad \lambda = -6, -1$$

30. A → r; B → q, s; C → p, t

(A) $\Delta'(x) = \begin{vmatrix} 2x-5 & 2x-5 & 3 \\ 6x+1 & 6x+1 & 9 \\ 14x-6 & 14x-6 & 21 \end{vmatrix}$

$$+ \begin{vmatrix} x^2 - 5x + 3 & 2 & 3 \\ 3x^2 + x + 4 & 6 & 9 \\ 7x^2 - 6x + 9 & 14 & 21 \end{vmatrix} + 0$$

$$= 3ax^2 + 2bx + c$$

$\therefore \quad \Delta'(0) =$ constant

$$\Rightarrow \qquad a = 0, b = 0, c = 0$$

and $\quad \Delta(0) = \begin{vmatrix} 3 & -5 & 3 \\ 4 & 1 & 9 \\ 9 & -6 & 21 \end{vmatrix} = d$

$$\Rightarrow \qquad d = 141$$

$$\therefore \qquad 3a + 4b + 5c + d = 0 + 0 + 0 + 141 = 141$$

(B) At $x = 0$, $\Delta(0) = \begin{vmatrix} -1 & 0 & 7 \\ -1 & -1 & 8 \\ 0 & 0 & 0 \end{vmatrix} = d$

$$\Rightarrow \qquad d = 0$$

and $\quad \Delta'(x) = \begin{vmatrix} 1 & 5x & 7 \\ 2x & x-1 & 8 \\ 2 & 3x & 0 \end{vmatrix} + \begin{vmatrix} x-1 & 5 & 7 \\ x^2-1 & 1 & 8 \\ 2x & 3 & 0 \end{vmatrix}$

$$= 3ax^2 + 2bx + c$$

$$\Delta'(0) = \begin{vmatrix} 1 & 0 & 7 \\ 0 & -1 & 8 \\ 2 & 0 & 0 \end{vmatrix} + \begin{vmatrix} -1 & 5 & 7 \\ -1 & 1 & 8 \\ 0 & 3 & 0 \end{vmatrix} + 0 = c$$

$$\Rightarrow \qquad c = 17$$

and $\quad \Delta''(x) = \begin{vmatrix} 0 & 5x & 7 \\ 2 & x-1 & 8 \\ 0 & 3x & 0 \end{vmatrix} + \begin{vmatrix} 1 & 5 & 7 \\ 2x & 1 & 8 \\ 2 & 3 & 0 \end{vmatrix} + 0$

$$+ \begin{vmatrix} 1 & 5 & 7 \\ 2x & 1 & 8 \\ 2 & 3 & 0 \end{vmatrix} + 0 + 0 = 6ax + 2b$$

$$\Rightarrow \qquad a = 21, b = 42$$

$$\therefore \qquad b - c = 25$$

and $a + 2b + 3c + 5d = 21 + 84 + 51 + 0 = 156$

(C) $\Delta(x) = \begin{vmatrix} -6x^3 - 3x^2 + 14x & -x + 3 & 0 \\ 4x^3 - 7x & 3x + 2 & 1 \\ -5x^3 - 8x^2 + 21x & -8x - 7 & 0 \end{vmatrix}$

[applying $R_1 \to R_1 - 2R_2$ and $R_3 \to R_3 - 3R_2$]

$$= -43x^4 - 73x^3 + 46x^2 + 161x$$

$\therefore a = 0, \ b = 161, \ c = 46, \ d = -73, e = -43$

$$\therefore \qquad c - d = 119$$

and $3a + 2b + 5c + 5d = 0 + 322 + 230 + 365 = 187$

TARGET CBSE 2017

TEST DRIVE

Comprehensive Simulator Test Series for XII Board Exam.

BOARD EXAM

(PAPER ONE PREP UP)

Typical Questions with Crispy Solutions

FULL TEST 1

(COMPLETE SYLLABUS) WITH SOLUTIONS

INSTRUCTIONS

- Question Numbers 1-4 in **Section A** carrying 1 mark each.
- Question Numbers 5-12 in **Section B** carrying 2 marks each.
- Question Numbers 13-23 in **Section C** carrying 4 marks each.
- Question Numbers 24-29 in **Section D** carrying 6 marks each.

Section A

1. The cartesian equation of a line is $6x - 2 = 3y + 1 = 2z - 2$. Find the vector equation of the line.

2. Evaluate $\int \frac{2}{1 + \cos 2x}\, dx$. [Foreign 2012]

3. If $\begin{bmatrix} a+b & 2 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$, then find the value of $|a - b|$.

4. Find the slope of the tangent to the curve $y^2 = 2x^3 - x^2 + 3$ at (2, 3).

Section B

5. For the matrix $A = \begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix}$, verify that $(\text{adj } A)A = |A| I$.

6. Form the differential equation representing the family of curves $y = A\cos(x + B)$, where A and B are parameters.

7. Probabilities of solving a specific problem independently by A and B are $\frac{1}{3}$ and $\frac{1}{4}$, respectively. If both try to solve the problem independently, then find the probability that problem is solved.

8. If $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are two non-collinear unit vectors such that $|\hat{\mathbf{a}} \times \hat{\mathbf{b}}| = \frac{\sqrt{3}}{2}$, then find the value of $(2\hat{\mathbf{a}} - 5\hat{\mathbf{b}}) \cdot (3\hat{\mathbf{a}} + \hat{\mathbf{b}})$.

9. If $e^y(x + 1) = 1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

10. Find the intervals in which $f(x) = \frac{4x^2 + 1}{x}$ is increasing.

11. Differentiate $\sin^{-1}\left(\frac{2x}{1 + x^2}\right)$ w.r.t. $\tan^{-1} x$, $-1 < x < 1$.

12. A man 2 m high, walks at a uniform speed of 6 m per minute away from a lamp-post 5 m high. Find the rate at which the length of his shadow increases.

Section C

13. If $y = \cot^{-1}(\sqrt{\cos x}) - \tan^{-1}(\sqrt{\cos x})$, then prove that $\sin y = \tan^2 \frac{x}{2}$.

14. Show that

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a + b + c)(ab + bc + ca).$$

15. Evaluate $\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}\, dx$.

16. Prove that for any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, $[\mathbf{a} + \mathbf{b}\ \mathbf{b} + \mathbf{c}\ \mathbf{c} + \mathbf{a}] = 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$. [Delhi 2014]

17. Find the vector equation of the plane through the points $(2, 1, -1)$ and $(-1, 3, 4)$ and perpendicular to the plane $x - 2y + 4z = 10$. [AI 2013]

Or

Find the equation of the plane passing through the point $A(1, 2, 1)$ and perpendicular to the line joining the points $P(1, 4, 2)$ and $Q(2, 3, 5)$. Also, find the distance of this plane from the line $\frac{x+3}{2} = \frac{y-5}{-1} = \frac{z-7}{-1}$. [Delhi 2010 C]

18. Solve the initial value problem

$y dx - (x + 2y^2) dy = 0,\ y(1) = 1.$

Or

Solve the following initial value problem

$\left(xe^{\frac{y}{x}} + y\right) dx = x dy,\ y(1) = 1.$

19. Evaluate $\int_{-\pi/4}^{\pi/4} \frac{\pi}{4(2 - \cos 2x)}\, dx.$

20. A clever student biased a die in such a way that an even number is twice likely to occur as an odd number. If the die is tossed four times and getting a perfect square is considered a success. What is the probability of atleast 3 success? Is this a good tendency? Justify your answer.

21. Find the value of a, for which the function f is defined as $f(x) = \begin{cases} a \sin \frac{\pi}{2}(x+1), & x \le 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$ is continuous at $x = 0$. [Delhi 2011]

22. Evaluate $\int e^{2x}\left(\frac{1 - \sin 2x}{1 - \cos 2x}\right) dx.$ [Delhi 2012 C]

Or

Evaluate $\int x \sin^{-1} x\, dx.$

23. The bag A contains 8 white and 7 black balls while the bag B contains 5 white and 4 black balls. One ball is randomly picked up from the bag A and mixed up with the balls in bag B. Then, a ball is randomly drawn out from it. Find the probability that ball drawn is white.

Section D

24. A dietician wishes to mix two types of food in such a way that the vitamin contents of the mixture contain atleast 8 units of vitamin A and 10 units of vitamin C. Food I contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C while food II contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs ₹ 5/kg to purchase food I and ₹ 7/kg to purchase food II. Determine the minimum cost to such a mixture. Formulate the above as LPP and solved it.

25. A manufacturer can sell x items at a price of ₹$\left(5 - \frac{x}{100}\right)$ each. The cost price of x items is ₹$\left(\frac{x}{5} + 500\right)$. Find the number of items he should sell to reach maximum profit. [AI 2009]

26. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0$, $y = 0$, $x = 4$ and $y = 4$ into three equal parts. [Delhi 2009]

Or

Using method of integration, find the area bounded by the curve $|x| + |y| = 1$.

27. Find the equation of the plane passing through the point $(1, 1, 1)$ and containing the line $\mathbf{r} = (-3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}}) + \lambda(3\hat{\mathbf{i}} - \hat{\mathbf{j}} - 5\hat{\mathbf{k}})$. Also, find the distance between plane and point $(6, 5, 9)$.

28. Using elementary operations, find the inverse of the matrix $A = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$.

Or

Solve the following system of equations by matrix method.

$$4x - 5y - 11z = 12$$
$$x - 3y + z = 1$$
$$2x + 3y - 7z = 2$$

29. Let $A = Q \times Q$ and '*' be a binary operation on A defined by $(a, b) * (c, d) = (ac, bc + d)$ for $(a, b), (c, d) \in A$, then with respect to * on A,

(i) find the identity element in A.

(ii) find the invertible elements of A.

Do You Know?

The Cursed 528th Digit Mathematician William Shanks calculated the value of pi (π) to 707 places but made a mistake on the 528th digit and thereby calculating every digit after it incorrectly. Pi (π) is not capable of being expressed as a fraction, thus making it an irrational number. It neither repeats nor does it end when written as a decimal.

Answers with Explanation

1. The equation of given line is

$$6x - 2 = 3y + 1 = 2z - 2$$

$$\Rightarrow \quad 6\left(x - \frac{1}{3}\right) = 3\left(y + \frac{1}{3}\right) = 2(z - 1)$$

$$\Rightarrow \quad \frac{x - \frac{1}{3}}{1} = \frac{y + \frac{1}{3}}{2} = \frac{z - 1}{3}$$

This shows that, the given line passes through $\left(\frac{1}{3}, -\frac{1}{3}, 1\right)$ and has direction ratios proportional to 1, 2, 3.

$\therefore$ Its vector equation is

$$\mathbf{r} = \left(\frac{1}{3}\hat{\mathbf{i}} - \frac{1}{3}\hat{\mathbf{j}} + \hat{\mathbf{k}}\right) + \lambda(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$

2. Let $I = \int \frac{2}{1 + \cos 2x}\, dx$

$$= \int \frac{2}{2\cos^2 x}\, dx \qquad [\because \cos 2\theta = 2\cos^2\theta - 1]$$

$$= \int \sec^2 x\, dx = \tan x + C$$

3. Given, $\begin{bmatrix} a + b & 2 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$

Since, the corresponding elements of two equal matrices are equal.

$\therefore \quad a + b = 6$ and $ab = 8$

Hence, $|a - b| = \sqrt{(a + b)^2 - 4ab}$

$$= \sqrt{(6)^2 - 4 \times 8} = 2$$

4. Given, $y^2 = 2x^3 - x^2 + 3$

On differentiating both sides w.r.t. x, we get

$$2y\frac{dy}{dx} = 6x^2 - 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2 - x}{y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(2,\,3)} = \frac{3(2)^2 - 2}{3} = \frac{10}{3}$$

$\therefore$ Slope of tangent to the curve is $\frac{10}{3}$.

5. Given, $A = \begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix}$

Cofactors of A are

$$C_{11} = 4,\ C_{12} = -2,$$

$$C_{21} = -5,\ C_{22} = -3$$

$$\therefore \quad \text{adj}\,(A) = \begin{bmatrix} 4 & -2 \\ -5 & -3 \end{bmatrix}^T = \begin{bmatrix} 4 & -5 \\ -2 & -3 \end{bmatrix}$$

Also, $|A| = -12 - 10 = -22$

Now, $(\text{adj } A)\ A = \begin{bmatrix} 4 & -5 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -22 & 0 \\ 0 & -22 \end{bmatrix}$

$$= -22\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= |A|I \qquad [\because |A| = -22]$$

6. Given, $y = A\cos(x + B)$ ...(i)

Since, the given equation has two arbitrary constants. We shall differentiate it two times.

On differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = -A\sin(x + B)$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -A\cos(x + B)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -y \Rightarrow \frac{d^2y}{dx^2} + y = 0$$

7. Let $P(A)$ = Probability that A solves the problem

$P(B)$ = Probability that B solves the problem

Then, $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$

The problem will be solved, if atleast one of them will solve it.

$\therefore$ Required probability

$$P(A \cup B)$$

$$= 1 - P(\overline{A})\ P(\overline{B})$$

$$= 1 - \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)$$

$$= 1 - \frac{2}{3} \times \frac{3}{4} = 1 - \frac{1}{2} = \frac{1}{2}$$

8. For any two unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$,

$$\left|\hat{\mathbf{a}} \times \hat{\mathbf{b}}\right|^2 + \left|\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}\right|^2 = |\hat{\mathbf{a}}|^2|\hat{\mathbf{b}}|^2$$

$$\Rightarrow \quad \left(\frac{\sqrt{3}}{2}\right)^2 + (\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})^2 = (1)^2 \cdot (1)^2 \qquad [\because |\hat{\mathbf{a}}| = |\hat{\mathbf{b}}| = 1]$$

$$\Rightarrow \quad (\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})^2 = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\Rightarrow \quad \hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = \frac{1}{2}$$

$$\therefore (2\hat{\mathbf{a}} - 5\hat{\mathbf{b}}) \cdot (3\hat{\mathbf{a}} + \hat{\mathbf{b}}) = 6|\hat{\mathbf{a}}|^2 - 13(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}) - 5|\hat{\mathbf{b}}|^2$$

$$= 6 \times 1 - 13 \times \frac{1}{2} - 5$$

$$= \frac{-11}{2}$$

9. Given, $e^y(x + 1) = 1$

$$\Rightarrow \quad e^y = \frac{1}{x + 1}$$

On taking log both sides, we get

$$\log e^y = \log\left(\frac{1}{x + 1}\right)$$

$$\Rightarrow \quad y = -\log(x + 1)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{x + 1}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{(x + 1)^2} = \left[\frac{-1}{x + 1}\right]^2$$

$$\therefore \quad \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 \qquad \textbf{Hence proved.}$$

10. Given, $f(x) = \dfrac{4x^2+1}{x}$

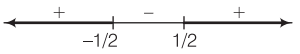
$\Rightarrow \quad f'(x) = \dfrac{4x^2-1}{x^2}$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0 \Rightarrow \frac{4x^2-1}{x^2} > 0$$

$\Rightarrow \quad 4x^2 - 1 > 0 \Rightarrow (2x-1)(2x+1) > 0$

$\Rightarrow \quad \left(x-\dfrac{1}{2}\right)\left(x+\dfrac{1}{2}\right) > 0 \Rightarrow x \in \left(-\infty, \dfrac{-1}{2}\right) \cup \left(\dfrac{1}{2}, \infty\right)$



Hence, $f(x)$ is increasing on $\left(-\infty, \dfrac{-1}{2}\right) \cup \left(\dfrac{1}{2}, \infty\right)$.

11. Let $u = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ and $v = \tan^{-1} x$

Putting $x = \tan\theta$, we get

$$u = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$$

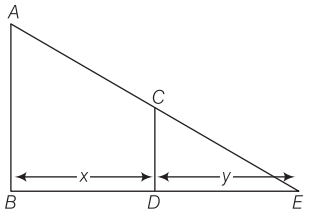
Since $\quad -1 < x < 1$

$\Rightarrow \quad \dfrac{-\pi}{4} < \theta < \dfrac{\pi}{4} \Rightarrow \dfrac{-\pi}{2} < 2\theta < \dfrac{\pi}{2}$

$\therefore \quad u = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1} x$

$\therefore \quad \dfrac{du}{dv} = \dfrac{\dfrac{du}{dx}}{\dfrac{dv}{dx}} = \dfrac{\dfrac{d}{dx}(2\tan^{-1} x)}{\dfrac{d}{dx}(\tan^{-1} x)} = 2$

12. Let AB be a lamp-post. Let at any time t, the man CD be at a distance of x m from the lamp-post and y m be the length of his shadow DE.



Clearly, ΔABE and ΔCDE are similar.

$\therefore \quad \dfrac{AB}{CD} = \dfrac{BE}{DE}$

$\Rightarrow \quad \dfrac{5}{2} = \dfrac{x+y}{y}$

$\Rightarrow \quad 3y = 2x$

$\Rightarrow \quad 3\dfrac{dy}{dt} = 2\dfrac{dx}{dt}$

$\Rightarrow \quad \dfrac{dy}{dt} = \dfrac{2}{3} \times 6 = 4 \qquad \left[\because \dfrac{dx}{dt} = 6 \text{ m/min}\right]$

Thus, shadow increases at the rate of 4 m/min.

13. $y = \cot^{-1}(\sqrt{\cos x}) - \tan^{-1}(\sqrt{\cos x})$

$\Rightarrow \quad y = \dfrac{\pi}{2} - \tan^{-1}(\sqrt{\cos x}) - \tan^{-1}(\sqrt{\cos x})$

$\Rightarrow \quad y = \dfrac{\pi}{2} - 2\tan^{-1}(\sqrt{\cos x})$

$\Rightarrow \quad y = \dfrac{\pi}{2} - \cos^{-1}\left\{\dfrac{1-(\sqrt{\cos x})^2}{1+(\sqrt{\cos x})^2}\right\}$

$\Rightarrow \quad y = \dfrac{\pi}{2} - \cos^{-1}\left\{\dfrac{1-\cos x}{1+\cos x}\right\}$

$\Rightarrow \quad y = \dfrac{\pi}{2} - \cos^{-1}\left(\tan^2\dfrac{x}{2}\right)$

$\Rightarrow \quad \cos^{-1}\left(\tan^2\dfrac{x}{2}\right) = \dfrac{\pi}{2} - y$

$\Rightarrow \quad \tan^2\dfrac{x}{2} = \cos\left(\dfrac{\pi}{2} - y\right) = \sin y$ **Hence proved.**

14. Let $\quad \Delta = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

$\Rightarrow \quad \Delta = (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Delta = (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & -b+a \\ 0 & -c+a & 2c+a \end{vmatrix}$$

Expanding along C_1,

$$\Delta = (a+b+c)\begin{vmatrix} 2b+a & -b+a \\ -c+a & 2c+a \end{vmatrix}$$

$\Rightarrow \quad \Delta = (a+b+c)\{(4bc + 2ab + 2ca + a^2) - (bc - ab - ac + a^2)\}$

$= (a+b+c)(3bc + 3ab + 3ca)$

$= 3(a+b+c)(ab + bc + ca)$ **Hence proved.**

15. Let $I = \displaystyle\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}\,dx$

Put $\quad x = t^6 \Rightarrow dx = 6t^5 dt$

$\therefore \quad I = \displaystyle\int \frac{6t^5}{t^3+t^2}\,dt = 6\int \frac{t^3}{t+1}\,dt = 6\int \frac{(t^3+1)-1}{(t+1)}\,dt$

$= 6\displaystyle\int\left(t^2 - t + 1 - \frac{1}{(t+1)}\right)dt$

$= 6\left[\dfrac{t^3}{3} - \dfrac{t^2}{2} + t - \log|t+1|\right] + C$

$= 2\sqrt{x} - 3x^{1/3} + 6x^{1/6} - 6\log|x^{1/6} + 1| + C$

16. LHS $= [\mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}]$

$= (\mathbf{a}+\mathbf{b})\cdot[(\mathbf{b}+\mathbf{c}) \times (\mathbf{c}+\mathbf{a})]$

$= (\mathbf{a}+\mathbf{b})\cdot[\mathbf{b}\times\mathbf{c} + \mathbf{b}\times\mathbf{a} + \mathbf{c}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}]$

$= (\mathbf{a}+\mathbf{b})\cdot[\mathbf{b}\times\mathbf{c} + \mathbf{b}\times\mathbf{a} + \mathbf{c}\times\mathbf{a}] \qquad [\because \mathbf{c}\times\mathbf{c} = 0]$

$= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) + \mathbf{a}\cdot(\mathbf{b}\times\mathbf{a}) + \mathbf{a}\cdot(\mathbf{a}\times\mathbf{a})$
$+ \mathbf{b}\cdot(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\cdot(\mathbf{b}\times\mathbf{a}) + \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$

$= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{a}\ \mathbf{b}\ \mathbf{a}] + [\mathbf{a}\ \mathbf{c}\ \mathbf{a}] + [\mathbf{b}\ \mathbf{b}\ \mathbf{c}]$
$+ [\mathbf{b}\ \mathbf{b}\ \mathbf{a}] + [\mathbf{b}\ \mathbf{c}\ \mathbf{a}]$

$= 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \text{RHS}$

Hence proved.

17. The equation of any plane through $(-1, 3, 4)$ is

$$a(x+1) + b(y-3) + c(z-4) = 0 \quad \ldots(i)$$

If it passes through $(2, 1, -1)$, then

$$a(2+1) + b(1-3) + c(-1-4) = 0$$

$$\Rightarrow \quad 3a - 2b - 5c = 0 \quad \ldots(ii)$$

If plane (i) is perpendicular to the plane $x - 2y + 4z = 10$, then

$$a - 2b + 4c = 0 \quad \ldots(iii)$$

On solving Eqs. (ii) and (iii), by cross-multiplication method, we get

$$\frac{a}{18} = \frac{b}{17} = \frac{c}{4} = \lambda \quad \text{[say]}$$

$$\Rightarrow \quad a = 18\lambda,\ b = 17\lambda \text{ and } c = 4\lambda$$

On putting the values of a, b and c in Eq. (i), we get

$$18\lambda(x+1) + 17\lambda(y-3) + 4\lambda(z-4) = 0$$

$$\Rightarrow \quad 18x + 17y + 4z = 49$$

$\therefore$ Vector equation is $\mathbf{r} \cdot (18\hat{\mathbf{i}} + 17\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = 49$

Or

The direction ratios of PQ are proportional to

$2-1, 3-4, 5-2$, i.e. $1, -1, 3$

So, direction ratios of plane perpendicular to the line PQ are $1, -1, 3$.

Thus, equation of plane passing through $(1, 2, 1)$ and perpendicular to PQ is

$$1 \times (x-1) + (-1)(y-2) + 3(z-1) = 0$$

$$\Rightarrow \quad x - y + 3z = 2 \quad \ldots(i)$$

Now, the given equation of line is

$$\frac{x+3}{2} = \frac{y-5}{-1} = \frac{z-7}{-1}$$

The distance between the plane (i) and the given line is distance of any point on the line from the plane (i).

So, required distance = Length of perpendicular from $(-3, 5, 7)$ on plane (i)

$$= \left| \frac{-3 - 5 + 21 - 2}{\sqrt{1^2 + (-1)^2 + 3^2}} \right| = \sqrt{11} \text{ units}$$

18. We have, $ydx - (x + 2y^2)dy = 0$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x + 2y^2}{y} \Rightarrow \frac{dx}{dy} + \left(\frac{-1}{y}\right)x = 2y$$

This is a linear differential equation of the form

$$\frac{dx}{dy} + Rx = S$$

$$\therefore \quad \text{IF} = e^{\int \left(-\frac{1}{y}\right)dy} = e^{-\log y} = y^{-1}$$

The solution of differential equation is

$$x \times (\text{IF}) = \int 2y \times (\text{IF})dy + C$$

$$x \times \frac{1}{y} = \int 2y \times \frac{1}{y}dy + C \Rightarrow \frac{x}{y} = 2y + C$$

Given that $y = 1$ when $x = 1$

$$\Rightarrow \quad C = -1$$

$$\therefore \quad \frac{x}{y} = 2y - 1 \Rightarrow x = y(2y-1)$$

Or

We have, $(xe^{\frac{y}{x}} + y)dx = xdy$

$$\Rightarrow \quad \frac{dy}{dx} = e^{\frac{y}{x}} + \frac{y}{x}$$

On putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$$v + x\frac{dv}{dx} = e^v + v$$

$$\Rightarrow \quad x\frac{dv}{dx} = e^v$$

$$\Rightarrow \quad \int e^{-v}dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad -e^{-v} = \log|x| + C$$

$$\Rightarrow \quad -e^{-\frac{y}{x}} = \log|x| + C$$

Given that $y = 1$ when $x = 1$

$$\Rightarrow \quad C = -\frac{1}{e}$$

$$\therefore \quad -e^{-\frac{y}{x}} = \log|x| - \frac{1}{e}$$

$$\Rightarrow \quad -\frac{y}{x} = \log(1 - e\log|x|) - 1$$

$$\Rightarrow \quad y = x - x\log(1 - e\log|x|)$$

19. Let $I = \int_{-\pi/4}^{\pi/4} \frac{\pi}{4(2 - \cos 2x)}dx$

$$= 2 \times \frac{\pi}{4}\int_0^{\pi/4} \frac{1}{2 - \cos 2x}dx$$

$$\left[\because \frac{1}{2 - \cos 2x} \text{ is an even function}\right]$$

$$= \frac{\pi}{2}\int_0^{\pi/4} \frac{1 + \tan^2 x}{1 + 3\tan^2 x}dx \quad \left[\because \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}\right]$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi/4} \frac{\sec^2 x}{1^2 + (\sqrt{3}\tan x)^2}dx$$

Put $\sqrt{3}\tan x = t$

$$\Rightarrow \quad \sqrt{3}\sec^2 \times dx = dt$$

$$\therefore \quad I = \frac{\pi}{2\sqrt{3}}\int_0^{\sqrt{3}} \frac{dt}{1 + t^2} = \frac{\pi}{2\sqrt{3}}[\tan^{-1}(t)]_0^{\sqrt{3}}$$

$$= \frac{\pi}{2\sqrt{3}} \times \frac{\pi}{3} = \frac{\pi^2}{6\sqrt{3}}$$

20. Let p_1 be the probability of getting an odd number and the probability of getting an even number is $2p_1$.

$$\therefore P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$$

$$\Rightarrow \quad p_1 + 2p_1 + p_1 + 2p_1 + p_1 + 2p_1 = 1$$

$$\Rightarrow \quad p_1 = \frac{1}{9}$$

Now, probability of getting a perfect square

$$p = P(1) + P(4) = p_1 + 2p_1$$

$$= 3p_1 = \frac{1}{3}$$

$$\therefore \quad q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$$

Let x denotes the number of success in 4 throws to die.

$\therefore$ Probability of atleast 3 success $= P(X \geq 3)$

$$= P(X = 3) + P(X = 4)$$

$$= {}^4C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^1 + {}^4C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^0 = \frac{8}{81} + \frac{1}{81} = \frac{1}{9}$$

No, it may be good once or twice but not forever.

Honesty pays in a long run.

To be Continued at Page 92

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- Question Numbers 13-23 in **Section C** are Long Answers I carrying 4 marks each.
- Question Numbers 24-29 in **Section D** are Long Answers II carrying 6 marks each.

Section A

1. Find a matrix X such that $3A - 2B + X = 0$, where $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$.

2. For what value of k, the tangent to the curve $y = kx^2 - 6x - 9$ at point $\left(\frac{3}{2}, \frac{-17}{2}\right)$ is parallel to the axis?

3. Evaluate $\int \frac{x^4 + x^2 - 1}{x^2 - x + 1}\, dx$.

4. Find the equation of the line passing through the point $A(-3, 2, 0)$ and parallel to the line joining the points $B(-1, 0, 7)$ and $C(4, -3, 5)$.

Section B

5. Find the magnitude of $\angle BAC$, where A, B, C are the vertices of ΔABC having position vectors as the points (2, 1, 3), (4, 1, 3) and (0, 2, 1), respectively.

6. The probability that a certain person will buy a shirt is 0.2, the probability that he will buy a trouser is 0.3 and the probability that he will buy a shirt given that he buys a trouser is 0.4. Find the probability that he will buy a trouser given that he buys a shirt.

7. Verify Rolle's theorem for the function $f(x) = x(x-3)^2$, $0 \le x \le 3$.

8. Find the approximate value of $f(3.02)$, where $f(x) = 3x^2 + 5x + 3$.

9. If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, then prove that $\frac{dy}{dx} = \frac{-y}{x} \cdot \frac{\log x}{\log y}$.

10. Show that the differential equation that represents two family of all parabolas having their axis of symmetry coincident with X-axis is $yy_2 + y_1^2 = 0$.

11. If $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$, then prove that $|AB| = |BA|$.

12. Show that $f(x) = x^2$ is differentiable at $x = 1$.

Section C

13. Solve for x, $\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3}$, $-1 < x < 1$. [Delhi 2011C]

14. Evaluate $\int_2^3 \frac{x^2 + x + 1}{x^2 - x}\, dx$.

15. Find a unit vector perpendicular to both of the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$, where $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. [Foreign 2014]

Or

Show that the four points having position vectors $6\hat{\mathbf{i}} - 7\hat{\mathbf{j}}$, $16\hat{\mathbf{i}} - 19\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$, $3\hat{\mathbf{i}} - 6\hat{\mathbf{k}}$, $2\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 10\hat{\mathbf{k}}$ are coplanar.

16. Find the particular solution of differential equation $x\frac{dy}{dx} + y = x \log x$, given that $y(1) = \frac{1}{4}$.

17. Find the shortest distance between the lines $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$, $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$. [Foreign 2014]

18. In a multiple choice examination with three possible answers (out of which only one is correct) for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing? [Delhi 2010, AI 2009]

Or

Find the probability distribution of number of doublets in three tosses of a pair of dice. [Delhi 2010C]

19. Solve the following determinant equation

$$\begin{vmatrix} 1 & x & 1 \\ p+1 & p+x & p+1 \\ 3 & x+2 & x+1 \end{vmatrix} = 0.$$

20. Show that the function $f(x)$ defined by

$$f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, & x > 0 \\ \dfrac{4(1-\sqrt{1-x})}{x}, & x < 0 \\ 2, & x = 0 \end{cases}$$

is continuous at $x = 2$. [AI 2009C]

Or

Find the value of k, if $f(x)$ is continuous at $x = 0$, where

$$f(x) = \begin{cases} \dfrac{1-\cos kx}{x \sin x}, & x \neq 0 \\ \dfrac{1}{2}, & x = 0 \end{cases}.$$

21. Evaluate $\int x^2 \tan^{-1} x \, dx$.

22. Evaluate $\int \dfrac{x}{(x+1)(x^2+1)} dx$.

23. Ram, Jai and Ria are over weight persons and are members of a cycling club that holds a time trial each Sunday during the summer. The independent probabilities that Ram, Jai and Ria take part in anyone of these trials are 0.6, 0.7 and 0.8 respectively. Find the probability that on a particular Sunday.

(i) none of the three cyclist takes part.

(ii) either one or two of the three cyclist take part.

How can cycling help in reducing weight of a person?

Section D

24. A manager need to buy two types of filled cabinets A and B for his office. The costs and limitations for the cabinets are as follows

Cabinet	Cost of each unit	Area occupied	Storage volume
X	10	6 m^2	8 m^3
Y	20	8 m^2	12 m^3

The office has room for cabinets no more than 72 sq m. How many of which model should we buy in order to maximise storage volume?

25. Find the image of the point (1, 6, 3) on the line $\dfrac{x}{1} = \dfrac{y-1}{2} = \dfrac{z-2}{3}$. Also, write the equation of the line joining the given points and its image and find the length of segment joining given point and its image. [Delhi 2010C]

26. Show that the area of right angled triangle of given hypotenuse is maximum when the triangle is isosceles.

Or

Find the shortest distance of the point $(0, c)$ from the parabola $y = x^2$, where $0 \le c \le 5$.

27. Using method of integration, find the area of the region bounded by the lines $y = 4x + 5$, $y = 5 - x$ and $4y = x + 5$.

28. Solve the following system of equations by matrix method.

$\dfrac{2}{x} - \dfrac{3}{y} + \dfrac{3}{z} = 10$, $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 10$, $\dfrac{3}{x} - \dfrac{1}{y} + \dfrac{2}{z} = 13$ [CBSE 2007]

Or

If $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$ are two square matrices, then find BA and hence solve the system of linear equations

$x - y + z = 4$, $x - 2y - 2z = 9$ and $2x + y + 3z = 1$.

29. Show that the relation R on the set $A = \{x \in Z : 0 \le x \le 12\}$ given by $R = \{(a, b) : |a - b| \text{ is a multiple of } 4\}$ is an equivalence relation. Find the set of all elements related to 1. [AI 2010]

Answers

1. $\begin{bmatrix} -16 & -4 \\ 3 & -5 \end{bmatrix}$ **2.** $k = 2$ **3.** $\dfrac{x^3}{3} + \dfrac{x^2}{2} + x + C$

4. $\dfrac{x-3}{-5} = \dfrac{y-2}{-3} = \dfrac{z}{-2}$ **5.** $\pi - \cos^{-1}\left(\dfrac{2}{3}\right)$

6. 0.6 **8.** 45.46 **13.** $x = 2 - \sqrt{3}$

14. $1 + \log\left(\dfrac{16}{3}\right)$ **15.** $\pm \dfrac{1}{\sqrt{6}}(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$

16. $y = \dfrac{1}{2}x \log x - \dfrac{x}{4} + \dfrac{1}{2x}$ **17.** $\sqrt{116}$ units

18. $\dfrac{11}{243}$ *Or*

x	0	1	2	3
$P(x)$	$\dfrac{125}{216}$	$\dfrac{75}{216}$	$\dfrac{15}{216}$	$\dfrac{1}{216}$

19. $x = 1, 2$ **20.** *Or* $k = \pm 1$

21. $\dfrac{x^3}{3}\tan^{-1} x - \dfrac{1}{6}x^2 + \dfrac{1}{6}\log|x^2+1| + C$

22. $\dfrac{-1}{2}\log|x+1| + \dfrac{1}{4}\log|x^2+1| + \dfrac{1}{2}\tan^{-1} x + C$

23. (i) 0.024 (ii) 0.64 **24.** Cabinet $X = 8$, Cabinet $Y = 3$

25. $(1, 0, 7)$, $\dfrac{x-1}{0} = \dfrac{y-6}{3} = \dfrac{z-3}{-2}$, $\sqrt{52}$ units **26.** *Or* $\dfrac{\sqrt{4c-1}}{2}$

27. $\dfrac{15}{2}$ sq units

28. $x = \dfrac{1}{2}, y = \dfrac{1}{3}, z = \dfrac{1}{5}$

Or $BA = 8Ix = 3$, $y = -2$ and $z = -1$

29. {1, 5, 9}

INTEGRATION

A collection of best questions asked in JEE to revise your concepts

Single Correct Questions

1. The value of $\int \frac{dx}{x^2\,(x^4+1)^{3/4}}$ is [2015]

(a) $\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}}+C$ (b) $(x^4+1)^{\frac{1}{4}}+C$

(c) $-(x^4+1)^{\frac{1}{4}}+C$ (d) $-\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}}+C$

2. $\int \frac{\sec^2 x}{(\sec x+\tan x)^{9/2}}\,dx$ is equal to (for some arbitrary constant K) [2012]

(a) $\frac{-1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(b) $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(c) $\frac{-1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(d) $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

3. $\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}\,dx$ is equal to [2014]

(a) $(x-1)\,e^{x+\frac{1}{x}}+C$ (b) $xe^{x+\frac{1}{x}}+C$

(c) $(x+1)e^{x+\frac{1}{x}}+C$ (d) $-xe^{x+\frac{1}{x}}+C$

4. If $\int f(x)\,dx=\psi(x)$, then $\int x^5 f(x^3)\,dx$ is equal to [2013]

(a) $\frac{1}{3}[x^3\psi(x^3)-\int x^2\psi(x^3)\,dx]+C$

(b) $\frac{1}{3}x^3\psi(x^3)-3\int x^3\psi(x^3)\,dx+C$

(c) $\frac{1}{3}x^3\psi(x^3)-\int x^2\psi(x^3)\,dx+C$

(d) $\frac{1}{3}[x^3\psi(x^3)-\int x^3\psi(x^3)\,dx]+C$

5. If $I=\int \frac{e^x}{e^{4x}+e^{2x}+1}\,dx$ and $J=\int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1}\,dx$.

Then, for any arbitrary constant C, the value of $J-I$ equals [2008]

(a) $\frac{1}{2}\log\left|\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right|+C$

(b) $\frac{1}{2}\log\left|\frac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right|+C$

(c) $\frac{1}{2}\log\left|\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right|+C$

(d) $\frac{1}{2}\log\left|\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right|+C$

6. If $f(x)=\frac{x}{(1+x^n)^{1/n}}$ for $n\geq 2$ and $g(x)=\underbrace{(fofo\ldots of)}_{n\text{ times}}x$.

Then, $\int x^{n-2}g(x)\,dx$ equals [2007]

(a) $\frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}}+C$

(b) $\frac{1}{n-1}(1+nx^n)^{1-\frac{1}{n}}+C$

(c) $\frac{1}{n(n+1)}(1+nx^n)^{1+\frac{1}{n}}+C$

(d) $\frac{1}{n+1}(1+nx^n)^{1+\frac{1}{n}}+C$

7. The value of $\int \frac{(x^2-1)}{x^3\sqrt{2x^4-2x^2+1}}\,dx$ is [2006]

(a) $2\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}+C$ (b) $2\sqrt{2+\frac{2}{x^2}+\frac{1}{x^4}}+C$

(c) $\frac{1}{2}\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}+C$ (d) None of these

8. $\int \frac{x^2(1-\ln x)}{(\ln x)^4-x^4}\,dx$ is equal to [2015]

(a) $\frac{1}{2}\ln\left(\frac{x}{\ln x}\right)-\frac{1}{4}\ln(\ln x^2-x^2)+C$

(b) $\frac{1}{4}\ln\left(\frac{\ln x-x}{\ln x+x}\right)-\frac{1}{2}\tan^{-1}\left(\frac{\ln x}{x}\right)+C$

(c) $\frac{1}{4}\ln\left(\frac{\ln x+x}{\ln x-x}\right)+\frac{1}{2}\tan^{-1}\left(\frac{\ln x}{x}\right)+C$

(d) $\frac{1}{4}\ln\left(\frac{\ln x-x}{\ln x+x}\right)+\frac{1}{2}\tan^{-1}\left(\frac{\ln x}{x}\right)+C$

9. $\int \frac{mx^{m+2n-1} - nx^{n-1}}{x^{2m+2n} + 2x^{m+n} + 1}\,dx$ is equal to [2015]

(a) $\frac{-x^m}{x^{m+n}+1} + C$ (b) $\frac{x^n}{x^{m+n}+1} + C$

(c) $\frac{-x^n}{x^{m+n}+1} + C$ (d) $\frac{x^m}{x^{m+n}+1} + C$

10. If $\int x^{27}\,(6x^2 + 5x + 4)\,(x^2 + x + 1)^6\,dx = f(x) + C$, then $f(x)$ is equal to [2014]

(a) $\frac{1}{7}(x^6 + x^5 + x^4)^7$ (b) $\frac{1}{7}(6x^5 + 5x^4 + 4x^3)^7$

(c) $\frac{1}{7}(6x^6 + 5x^5 + 4x^4)^7$ (d) $\frac{1}{7}(x^5 + x^4 - x^3)^7$

11. $\int \sin(101\,x)\sin^{99}x\,dx$ equals [2014]

(a) $\frac{1}{100}\sin(100x)(\sin x)^{100} + C$

(b) $\frac{1}{100}\cos(100x)(\sin x)^{100} + C$

(c) $\frac{1}{100}\cos(100x)(\cos x)^{100} + C$

(d) $\frac{1}{100}\sin(100x)(\sin x)^{101} + C$

12. If $\int \frac{1 - 5\sin^2 x}{\cos^5 x \sin^2 x}\,dx = \frac{f(x)}{\cos^5 x} + C$, then $f(x)$ is [2014]

(a) $-\cot x$ (b) $-\operatorname{cosec} x$

(c) $\operatorname{cosec} x$ (d) $\cot x$

13. $\int \sqrt{x-3}\,\{\sin^{-1}(\ln x) + \cos^{-1}(\ln x)\}\,dx$ is equal to [2015]

(a) $\frac{\pi}{3}(x-3)^{3/2} + C$ (b) 0

(c) 1 (d) None of these

14. If $\int \frac{e^x\,(2 - x^2)}{(1-x)\sqrt{1-x^2}}\,dx = \mu \cdot e^x \left(\frac{1+x}{1-x}\right)^{\lambda} + C$, then the value of $2(\lambda + \mu)$ is [2013]

(a) 1 (b) 2

(c) 3 (d) 6

15. Let $f(x)$ be a quadratic function such that $f(0) = 1$ and $\int \frac{f(x)}{x^2(x+1)^3}\,dx$ is a rational function, then the value of $f'(0)$ is [2015]

(a) 0 (b) 2

(c) 3 (d) 5

16. If $\int f(x)\,dx = f(x)$, then $\int \{f(x)\}^2\,dx$ is equal to [2006]

(a) $\frac{1}{2}\{f(x)\}^2$ (b) $\{f(x)\}^3$

(c) $\frac{|f(x)|^3}{3}$ (d) $\{f(x)\}^2$

Analytical Questions

17. For any natural number m, evaluate $\int (x^{3m} + x^{2m} + x^m)\,(2x^{2m} + 3x^m + 6)^{1/m}\,dx, x > 0$. [2002]

18. Find the indefinite integral of $\int \left(\frac{1}{\sqrt[3]{x} + \sqrt[4]{x}} + \frac{\ln(1 + \sqrt[6]{x})}{\sqrt[3]{x} + \sqrt{x}}\right) dx$. [2000]

19. Evaluate $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2 + 8x + 13}}\right) dx$. [2004]

20. Evaluate $\int \frac{\sqrt{\cos 2x}}{\sin x}\,dx$. [2001]

Answers with Explanation

1. *(d)* $\int \frac{dx}{x^2(x^4+1)^{3/4}} = \int \frac{dx}{x^5\left(1 + \frac{1}{x^4}\right)^{3/4}}$

Put $1 + \frac{1}{x^4} = t^4$

$\Rightarrow -\frac{4}{x^5}\,dx = 4t^3\,dt$

$\Rightarrow \frac{dx}{x^5} = -t^3\,dt$

Hence, the integral becomes

$$\int \frac{-t^3}{t^3}\,dt = -\int dt = -t + C = -\left(1 + \frac{1}{x^4}\right)^{1/4} + C$$

$$= -\left(\frac{x^4+1}{x^4}\right)^{1/4} + C$$

2. *(c)* **PLAN** Integration by substitution,

i.e. $I = \int f(g(x))\,g'(x)\,dx$

Put $g(x) = t \Rightarrow g'(x)\,dx = dt$

$\therefore I = \int f(t)\,dt$

Also, if $\sec x + \tan x = t$

$\Rightarrow \sec x - \tan x = \frac{1}{t}$

Let $I = \int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}}\,dx$

Put $\sec x + \tan x = t$

$\Rightarrow (\sec x \tan x + \sec^2 x)\,dx = dt$

$\Rightarrow \sec x(\tan x + \sec x)\,dx = dt$

$\Rightarrow \sec x \cdot t\,dx = dt$

$\Rightarrow \sec x\,dx = \frac{1}{t}\,dt$

Also, $\sec x - \tan x = \frac{1}{t} \Rightarrow \sec x = \frac{1}{2}\left(t + \frac{1}{t}\right)$

$$\therefore\ I = \int \frac{\sec x \cdot \sec x}{(\sec x + \tan x)^{9/2}}\,dx = \int \frac{\frac{1}{2}\left(t + \frac{1}{t}\right)\cdot\frac{1}{t}\,dt}{t^{9/2}}$$

$$= \frac{1}{2}\int\left(\frac{1}{t^{9/2}} + \frac{1}{t^{13/2}}\right)dt = -\frac{1}{2}\left[\frac{2}{7t^{7/2}} + \frac{2}{11t^{11/2}}\right] + K$$

$$= -\left[\frac{1}{7(\sec x + \tan x)^{7/2}} + \frac{1}{11(\sec x + \tan x)^{11/2}}\right] + K$$

$$= \frac{-1}{(\sec x + \tan x)^{11/2}}\left[\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right] + K$$

3. ***(b)*** $\int\left(1 + x - \frac{1}{x}\right)e^{x + \frac{1}{x}}\,dx$

$$= \int e^{x+\frac{1}{x}}\,dx + \int x\left(1 - \frac{1}{x^2}\right)e^{x+\frac{1}{x}}\,dx$$

$$= \int e^{x+\frac{1}{x}}\,dx + xe^{x+\frac{1}{x}} - \int \frac{d}{dx}(x)\cdot e^{x+\frac{1}{x}}\,dx$$

$$= \int e^{x+\frac{1}{x}}\,dx + xe^{x+\frac{1}{x}} - \int e^{x+\frac{1}{x}}\,dx$$

$$\left[\because \int\left(1 - \frac{1}{x^2}\right)e^{x+\frac{1}{x}}\,dx = e^{x+\frac{1}{x}}\right]$$

$$= xe^{x+\frac{1}{x}} + C$$

4. ***(c)*** Given, $\int f(x)\,dx = \psi(x)$

Let $I = \int x^5 f(x^3)\,dx$

Put $x^3 = t \Rightarrow 3x^2\,dx = dt$

$\Rightarrow$ $x^2\,dx = \frac{1}{3}\,dt$ ...(i)

$\therefore$ $I = \frac{1}{3}\int t\,f(t)\,dt$

$$= \frac{1}{3}\left[t\int f(t)\,dt - \int\left\{\frac{d}{dt}(t)\cdot\int f(t)\,dt\right\}dt\right] + C$$

$$= \frac{1}{3}[t\psi(t) - \int \psi(f)\,dt] + C$$

$$= \frac{1}{3}[x^3\psi(x^3) - 3\int x^2\psi(x^3)\,dx] + C \text{ [from Eq. (i)]}$$

$$= \frac{1}{3}x^3\psi(x^3) - \int x^2\psi(x^3)\,dx + C$$

5. ***(c)*** Since, $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1}\,dx$

and $J = \int \frac{e^{3x}}{1 + e^{2x} + e^{4x}}\,dx$

$\therefore$ $J - I = \int \frac{e^{3x} - e^x}{e^{4x} + e^{2x} + 1}\,dx$

Put $e^x = t \Rightarrow e^x dx = dt$

$\therefore$ $J - I = \int \frac{t^2 - 1}{t^4 + t^2 + 1}\,dt$

$$= \int \frac{1 - \frac{1}{t^2}}{t^2 + 1 + \frac{1}{t^2}}\,dt = \int \frac{1 - \frac{1}{t^2}}{\left(t + \frac{1}{t}\right)^2 - 1}\,dt$$

Put $t + \frac{1}{t} = u \Rightarrow \left(1 - \frac{1}{t^2}\right)dt = du$

$\therefore$ $J - I = \int \frac{du}{u^2 - 1} = \frac{1}{2}\log\frac{u-1}{u+1} + C$

$$= \frac{1}{2}\log\left|\frac{t^2 - t + 1}{t^2 + t + 1}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1}\right| + C$$

6. ***(a)*** Given, $f(x) = \frac{x}{(1 + x^n)^{1/n}}$ for $n \geq 2$

$\therefore$ $ff(x) = \frac{f(x)}{[1 + f(x)^n]^{1/n}} = \frac{x}{(1 + 2x^n)^{1/n}}$

and $fff(x) = \frac{x}{(1 + 3x^n)^{1/n}}$

$\therefore$ $g(x) = \underbrace{(fofo...of)}_{n \text{ times}}(x) = \frac{x}{(1 + nx^n)^{1/n}}$

Let $I = \int x^{n-2}\,g(x)\,dx = \int \frac{x^{n-1}}{(1 + nx^n)^{1/n}}\,dx$

$$= \frac{1}{n^2}\int \frac{n^2x^{n-1}}{(1 + nx^n)^{1/n}}\,dx$$

$$= \frac{1}{n^2}\int \frac{\frac{d}{dx}(1 + nx^n)}{(1 + nx^n)^{1/n}}\,dx$$

$$= \frac{1}{n^2}\cdot\frac{(1 + nx^n)^{-\frac{1}{n}+1}}{\left(\frac{-1}{n} + 1\right)} + C$$

$$= \frac{1}{n(n-1)}(1 + nx^n)^{1-\frac{1}{n}} + C$$

7. ***(c)*** Let $I = \int \frac{(x^2 - 1)}{x^3\sqrt{2x^4 - 2x^2 + 1}}\,dx$

On dividing numerator and denominator by x^5, we get

$$I = \int \frac{\frac{1}{x^3} - \frac{1}{x^5}}{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}}\,dx$$

Put $2 - \frac{2}{x^2} + \frac{1}{x^4} = t \Rightarrow \left(\frac{4}{x^3} - \frac{4}{x^5}\right)dx = dt$

$\therefore$ $I = \frac{1}{4}\int \frac{dt}{\sqrt{t}} = \frac{1}{4}\cdot\frac{t^{1/2}}{1/2} + C = \frac{1}{2}\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}} + C$

8. ***(b)*** Let $I = \int \frac{x^2(1 - \ln x)}{(\ln x)^4 - x^4}\,dx$

$\Rightarrow$ $I = \int \frac{x^2(1 - \ln x)}{\{(\ln x)^2 + x^2\}(\ln x + x)(\ln x - x)}\,dx$

$$\Rightarrow I = \int \frac{\frac{1 - \ln x}{x^2}}{\left\{\left(\frac{\ln x}{x}\right)^2 + 1\right\}\left(\frac{\ln x}{x} + 1\right)\left(\frac{\ln x}{x} - 1\right)}\,dx$$

Put $\frac{\ln x}{x} = t \Rightarrow \frac{1 - \ln x}{x^2}\,dx = dt$

$$\therefore\ I=\int\frac{1}{(t^2+1)(t+1)(t-1)}\,dt=\int\frac{1}{(t^2+1)(t^2-1)}\,dt$$

$$=\frac{1}{2}\int\left(\frac{1}{t^2-1}-\frac{1}{t^2+1}\right)dt$$

$$=\frac{1}{4}\ln\left(\frac{t-1}{t+1}\right)-\frac{1}{2}\tan^{-1}t+C$$

$$\therefore\ I=\frac{1}{4}\ln\left(\frac{\frac{\ln x}{x}-1}{\frac{\ln x}{x}+1}\right)-\frac{1}{2}\tan^{-1}\left(\frac{\ln x}{x}\right)+C$$

$$=\frac{1}{4}\ln\left(\frac{\ln x-x}{\ln x+x}\right)-\frac{1}{2}\tan^{-1}\left(\frac{\ln x}{x}\right)+C$$

9. ***(c)*** Let $I=\int\frac{mx^{m+2n-1}-nx^{n-1}}{x^{2m+2n}+2x^{m+n}+1}\,dx$

$$=\int\frac{(mx^{m+n}-n)\,x^{n-1}}{(x^{m+n}+1)^2}\,dx$$

$$=\int\frac{(m+n)\,x^{m+2n-1}-nx^{m+2n-1}-nx^{n-1}}{(x^{m+n}+1)^2}\,dx$$

$$=\int\frac{(m+n)x^{m+n-1}\,x^n-nx^{n-1}(x^{m+n}+1)}{(x^{m+n}+1)^2}\,dx$$

$$=\int x^n\cdot\frac{(m+n)\,x^{m+n-1}}{(x^{m+n}+1)^2}\,dx-\int\frac{nx^{n-1}}{x^{m+n}+1}\,dx$$

$$=\frac{-x^n}{x^{m+n}+1}-\int nx^{n-1}\cdot\frac{-1}{x^{m+n}+1}\,dx-\int\frac{nx^{n-1}}{x^{m+n}+1}\,dx$$

$$=\frac{-x^n}{x^{m+n}+1}+\int\frac{nx^{n-1}}{x^{m+n}+1}-\int\frac{nx^{n-1}}{x^{m+n}+1}\,dx$$

$$=\frac{-x^n}{x^{m+n}+1}+C$$

10. ***(a)*** Let $I=\int x^{27}(6x^2+5x+4)\,(x^2+x+1)^6\,dx$

$$=\int(6x^5+5x^4+4x^3)\,(x^6+x^5+x^4)^6\,dx$$

$$=\int(x^6+x^5+x^4)^6\,d(x^6+x^5+x^4)$$

$$=\frac{1}{7}(x^6+x^5+x^4)^7+C$$

$$\therefore\quad f(x)=\frac{1}{7}(x^6+x^5+x^4)^7$$

11. ***(a)*** Let $I=\int\sin(101x)\sin^{99}x\,dx$

$$=\int\sin(100x+x)\sin^{99}x\,dx$$

$$=\int\underbrace{\sin 100x}_{\text{I}}\,\underbrace{\cos x\sin^{99}x}_{\text{II}}\,dx+\int\cos(100x)\sin^{100}x\,dx$$

$$=\sin(100x)\cdot\frac{\sin^{100}x}{100}-\int\cos(100x)\sin^{100}x\,dx+\int\cos(100x)\cdot\sin^{100}x\,dx$$

$$=\frac{1}{100}\sin(100x)\sin^{100}x+C$$

12. ***(d)*** Let $I=\int\frac{1-5\sin^2x}{\cos^5x\sin^2x}\,dx$

$$=\int\frac{(\cos^2x+\sin^2x)-5\sin^2x}{\cos^5x\sin^2x}\,dx$$

$$=\int\frac{\cos^2x-4\sin^2x}{\cos^5x\sin^2x}\,dx$$

$$=\int\frac{\cos^5x-4\sin^2x\cos^3x}{\cos^8x\sin^2x}\,dx$$

$$=\int\frac{d}{dx}\left(\frac{1}{\cos^4x\sin x}\right)$$

$$=\frac{1}{\cos^4x\sin x}+C$$

$$=\frac{\cot x}{\cos^5x}+C$$

$$\because\quad\frac{f(x)}{\cos^5x}+C=\frac{\cot x}{\cos^5x}+C$$

$$\therefore\quad f(x)=\cot x$$

13. ***(d)*** We observe that the integrand is $f(x)=\sqrt{x-3}\,\{\sin^{-1}(\ln x)+\cos^{-1}(\ln x)\}$ is not defined for any $x\in R$, i.e. its domain is ϕ. So, the given integral does not exist.

14. ***(c)*** Let $I=\int\frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}\,dx$

$$=\int e^x\left[\frac{\{1+(1-x^2)\}}{(1-x)\sqrt{1-x^2}}\right]dx$$

$$=\int e^x\left\{\frac{1}{(1-x)\sqrt{1-x^2}}+\sqrt{\frac{1+x}{1-x}}\right\}dx$$

$$=e^x\sqrt{\frac{1+x}{1-x}}+C$$

$$\therefore\quad\lambda=\frac{1}{2}\text{ and }\mu=1$$

Hence, $2(\lambda+\mu)=3$

15. ***(c)*** Let $g(x)=\int\frac{f(x)}{x^2(x+1)^3}\,dx$. It is given that $f(x)$ is a quadratic function.

So, let

$$\frac{f(x)}{x^2(x+1)^3}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}+\frac{E}{(x+1)^3}$$

$$\therefore\quad g(x)=\int\frac{f(x)}{x^2(x+1)^3}\,dx$$

$$\Rightarrow\quad g(x)=\int\left\{\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}+\frac{E}{(x+1)^3}\right\}dx \qquad \ldots(i)$$

$$\Rightarrow\quad g(x)=A\ln x-\frac{B}{x}+C\ln(x+1)-\frac{D}{(x+1)}-\frac{E}{2(x+1)^2}+K$$

It is given that $g(x)$ is a rational function.

$$\therefore\quad A=C=0$$

Putting $A=C=0$ in Eq. (i), we get

$$g(x)=\int\left\{\frac{B}{x^2}+\frac{D}{(x+1)^2}+\frac{E}{(x+1)^3}\right\}dx$$

$$\Rightarrow\quad g(x)=\int\left\{\frac{B(x+1)^3+Dx^2(x+1)+Ex^2}{x^2(x+1)^3}\right\}dx$$

$\therefore \quad f(x) = B(x+1)^3 + Dx^2(x+1) + Ex^2$

$\left[\because g(x) = \int \frac{f(x)}{x^2(x+1)^3}\right]$

$\Rightarrow \quad f(x) = (B+D)x^3 + (3B+D+E)x^2 + 3Bx + B$

But $f(x)$ is a quadratic function such that $f(0) = 1$.

Therefore, $B + D = 0$ and $B = 1 \Rightarrow D = -1$

$\therefore \quad f(x) = (2+E)x^2 + 3x + 1 \Rightarrow f'(x) = 2(2+E)x + 3$

$\therefore \quad f'(0) = 3$

16. ***(a)*** We have, $\int f(x)\,dx = f(x) \Rightarrow f(x) = \frac{d}{dx} f(x)$

$\Rightarrow \frac{1}{f(x)} d\{f(x)\} = dx \Rightarrow \log f(x) = x + \log C$

$\Rightarrow f(x) = Ce^x \Rightarrow \{f(x)\}^2 = C^2e^{2x}$

$\therefore \int \{f(x)\}^2\,dx = \int C^2e^{2x}dx = \frac{1}{2}C^2e^{2x} = \frac{1}{2}\{f(x)\}^2$

17. For any natural number m, the given integral can be written as

$I = \int (x^{3m} + x^{2m} + x^m)\frac{(2x^{3m} + 3x^{2m} + 6x^m)^{1/m}}{x}\,dx$

$= \int (2x^{3m} + 3x^{2m} + 6x^m)^{1/m}(x^{3m-1} + x^{2m-1} + x^{m-1})\,dx$

Put $2x^{3m} + 3x^{2m} + 6x^m = t$

$\Rightarrow (6mx^{3m-1} + 6mx^{2m-1} + 6mx^{m-1})\,dx = dt$

$\therefore \; I = \int t^{1/m}\frac{dt}{6m} = \frac{1}{6m}\cdot\frac{t^{\frac{1}{m}+1}}{\left(\frac{1}{m}+1\right)}$

$= \frac{1}{6(m+1)}\cdot(2x^{3m} + 3x^{2m} + 6x^m)^{(m+1)/m} + C$

18. Let $I = \int\left(\frac{1}{\sqrt[3]{x} + \sqrt[4]{x}} + \frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x} + \sqrt{x}}\right)dx$

$\therefore \quad I = I_1 + I_2$

where, $I_1 = \int\left(\frac{1}{\sqrt[3]{x} + \sqrt[4]{x}}\right)dx$,

$I_2 = \int \frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x} + \sqrt{x}}\,dx$

Now, $I_1 = \int\left(\frac{1}{\sqrt[3]{x} + \sqrt[4]{x}}\right)dx$

Put $x = t^{12} \Rightarrow dx = 12t^{11}dt$

$\therefore \quad I_1 = 12\int \frac{t^{11}}{t^4 + t^3}\,dt = 12\int \frac{t^8}{t+1}dt$

$= 12\int (t^7 - t^6 + t^5 - t^4 + t^3 - t^2 + t - 1)\,dt + 12\int \frac{dt}{t+1}$

$= 12\left(\frac{t^8}{8} - \frac{t^7}{7} + \frac{t^6}{6} - \frac{t^5}{5} + \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} - t\right) + 12\ln(t+1)$

and $I_2 = \int\left\{\frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x} + \sqrt{x}}\right\}dx$

Put $x = u^6 \Rightarrow dx = 6u^5\,du$

$\therefore \quad I_2 = \int \frac{\ln(1+u)}{u^2 + u^3}6u^5\,du = \int \frac{\ln(1+u)}{u^2(1+u)}\cdot 6u^5\,du$

$= 6\int \frac{u^3}{(u+1)}\ln(1+u)\,du$

$= 6\int\left(\frac{u^3 - 1 + 1}{u+1}\right)\ln(1+u)\,du$

$= 6\int\left(u^2 - u + 1 - \frac{1}{u+1}\right)\ln(1+u)\,du$

$= 6\int \underset{\text{II}}{(u^2 - u + 1)}\ln(1+u)\,du - 6\int \underset{\text{I}}{\frac{\ln(1+u)}{(u+1)}}\,du$

$= 6\left(\frac{u^3}{3} - \frac{u^2}{2} + u\right)\ln(1+u)$

$- \int \frac{2u^3 - 3u^2 + 6u}{u+1}\,du - 6\frac{1}{2}[\ln(1+u)]^2$

$= (2u^3 - 3u^2 + 6u)\ln(1+u)$

$- \int\left(2u^2 - 5u + \frac{11u}{u+1}\right)du - 3[\ln(1+u)]^2$

$= (2u^3 - 3u^2 + 6u)\ln(1+u)$

$- \left(\frac{2u^3}{3} - \frac{5}{2}u^2 + 11u - 11\ln(u+1)\right) - 3[\ln(1+u)]^2$

$\therefore \; I = \frac{3}{2}x^{2/3} - \frac{12}{7}x^{7/12} + 2x^{1/2} - \frac{12}{5}x^{5/12} + 3x^{1/3} - 4x^{1/4}$

$- 6x^{1/6} - 12x^{1/12} + 12\ln(x^{1/12} + 1)$

$+ (2x^{1/2} - 3x^{1/3} + 6x^{1/6})\ln(1 + x^{1/6})$

$- \left[\frac{2}{3}x^{1/2} - \frac{5}{2}x^{1/3}\,11x^{1/6} - 11\ln(1 + x^{1/6})\right]$

$- 3[\ln(1 + x^{1/6})]^2 + C$

$= \frac{3}{2}x^{2/3} - \frac{12}{7}x^{7/12} + \frac{4}{3}x^{1/2} - \frac{12}{5}x^{5/12}$

$+ \frac{1}{2}x^{1/3} - 4x^{1/4} - 7x^{1/6} - 12x^{1/12}$

$+ (2x^{1/2} - 3x^{1/3} + 6x^{1/6} + 11)\ln(1 + x^{1/6})$

$+ 12\ln(1 + x^{1/12}) - 3[\ln(1 + x^{1/6})]^2 + C$

19. Let $I = \int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2 + 8x + 13}}\right)dx$

$= \int \sin^{-1}\left(\frac{2x+2}{\sqrt{(2x+2)^2 + 9}}\right)dx$

Put $2x + 2 = 3\tan\theta \Rightarrow 2dx = 3\sec^2\theta\,d\theta$

$\therefore \quad I = \int \sin^{-1}\left(\frac{3\tan\theta}{\sqrt{9\tan^2\theta + 9}}\right)\cdot\frac{3}{2}\sec^2\theta\,d\theta$

$= \int \sin^{-1}\left(\frac{3\tan\theta}{3\sec\theta}\right)\cdot\frac{3}{2}\sec^2\theta\,d\theta$

$= \int \sin^{-1}\left(\frac{\sin\theta}{\cos\theta\cdot\sec\theta}\right)\cdot\frac{3}{2}\sec^2\theta\,d\theta$

$= \frac{3}{2}\int \sin^{-1}(\sin\theta)\cdot\sec^2\theta\,d\theta$

$= \frac{3}{2}\int \theta\cdot\sec^2\theta\,d\theta = \frac{3}{2}[\theta\cdot\tan\theta - \int 1\cdot\tan\theta\,d\theta]$

$= \frac{3}{2}[\theta\tan\theta - \log\sec\theta] + C$

$$= \frac{3}{2}\left[\tan^{-1}\left(\frac{2x+2}{3}\right)\cdot\left(\frac{2x+2}{3}\right) - \log\sqrt{1+\left(\frac{2x+2}{3}\right)^2}\right] + C_1$$

$$= (x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\log\left[1+\left(\frac{2x+2}{3}\right)^2\right] + C_1$$

$$= (x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\log(4x^2+8x+13) + C$$

$$\left[\text{let } \frac{3}{2}\log 3 + C_1 = C\right]$$

20. Let $I = \int \frac{\sqrt{\cos 2x}}{\sin x}\,dx = \int \sqrt{\frac{\cos^2 x - \sin^2 x}{\sin^2 x}}\,dx$

$$= \int \sqrt{\cot^2 x - 1}\,dx$$

Put $\cot x = \sec\theta \Rightarrow -\operatorname{cosec}^2 x\,dx = \sec\theta\tan\theta\,d\theta$

$$\therefore \quad I = \int \sqrt{\sec^2\theta - 1}\cdot\frac{\sec\theta\cdot\tan\theta}{-(1+\sec^2\theta)}\,d\theta$$

$$= -\int \frac{\sec\theta\cdot\tan^2\theta}{1+\sec^2\theta}\,d\theta = -\int\frac{\sin^2\theta}{\cos\theta+\cos^3\theta}\,d\theta$$

$$= -\int\frac{1-\cos^2\theta}{\cos\theta(1+\cos^2\theta)}\,d\theta$$

$$= -\int\frac{(1+\cos^2\theta)-2\cos^2\theta}{\cos\theta(1+\cos^2\theta)}\,d\theta$$

$$= -\int\sec\theta\,d\theta + 2\int\frac{\cos\theta}{1+\cos^2\theta}\,d\theta$$

$$= -\log|\sec\theta+\tan\theta| + 2\int\frac{\cos\theta}{2-\sin^2\theta}\,d\theta$$

$$= -\log|\sec\theta+\tan\theta| + \int\frac{dt}{2-t^2}, \text{ where } \sin\theta = t$$

$$= -\log|\sec\theta+\tan\theta| + 2\cdot\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+\sin\theta}{\sqrt{2}-\sin\theta}\right| + C$$

$$= -\log\left|\cot x + \sqrt{\cot^2 x - 1}\right| + \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right| + C$$

Continued from Page 84

21. $\text{LHL} = \lim\limits_{x\to 0^-} f(x)$

$$= \lim_{x\to 0^-} a\sin\frac{\pi}{2}(x+1) = \lim_{h\to 0} a\sin\frac{\pi}{2}(1-h)$$

$$= a\sin\frac{\pi}{2} = a$$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{\tan x - \sin x}{x^3}$$

$$= \lim_{h\to 0}\frac{\tan(h)-\sin(h)}{h^3} = \lim_{h\to 0}\frac{\sin h(1-\cos h)}{h^3\cos h}$$

$$= \lim_{h\to 0}\frac{\sin h}{h}\times\lim_{h\to 0}\frac{1-\cos h}{h^2}\times\lim_{h\to 0}\frac{1}{\cos h}$$

$$= \lim_{h\to 0}\frac{2\sin^2\frac{h}{2}}{\left(\frac{h}{2}\right)^2\cdot 4} = \frac{1}{2}$$

Since, the function is continuous at $x = 0$.

$$\therefore \qquad \text{LHL} = \text{RHL} \quad \Rightarrow \quad a = \frac{1}{2}$$

22. Let $I = \int e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$

$$= \int e^{2x}\left(\frac{1-2\sin x\cos x}{2\sin^2 x}\right)dx$$

$$= \frac{1}{2}\int e^{2x}(\operatorname{cosec}^2 x - 2\cot x)\,dx$$

$$= \frac{1}{2}\int e^{2x}\operatorname{cosec}^2 x\,dx - \int e^{2x}\cdot\cot x\,dx$$

$$= \frac{1}{2}[-e^{2x}\cot x + \int 2e^{2x}\cot x\,dx] - \int e^{2x}\cdot\cot x\,dx + C$$

$$= -\frac{e^{2x}}{2}\cot x + \int e^{2x}\cot x\,dx - \int e^{2x}\cot x\,dx + C$$

$$= -\frac{e^{2x}}{2}\cot x + C$$

Or

Let $I = \int x\sin^{-1}x\,dx$

$$= (\sin^{-1}x)\frac{x^2}{2} - \int\frac{1}{\sqrt{1-x^2}}\cdot\frac{x^2}{2}dx$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\left[\int\frac{1-x^2}{\sqrt{1-x^2}}dx - \int\frac{1}{\sqrt{1-x^2}}dx\right]$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\left[\int\sqrt{1-x^2}\,dx - \int\frac{1}{\sqrt{1-x^2}}dx\right]$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2} - \frac{1}{4}\sin^{-1}x + C$$

23. Let E_1, E_2 and A be the events defined below

E_1 = ball drawn from bag A is white,

E_2 = ball drawn from bag A is black

and A = ball drawn from bag B is white

Then, $P(E_1) = \frac{8}{15}$, $P(E_2) = \frac{7}{15}$

$$P\left(\frac{A}{E_1}\right) = \frac{6}{10},$$

$$P\left(\frac{A}{E_2}\right) = \frac{5}{10}$$

By the law of total probability,

$$P(A) = P(E_1)\times P\left(\frac{A}{E_1}\right) + P(E_2)\times P\left(\frac{A}{E_2}\right)$$

$$= \frac{8}{15}\times\frac{6}{10} + \frac{7}{15}\times\frac{5}{10}$$

$$= \frac{83}{150}$$

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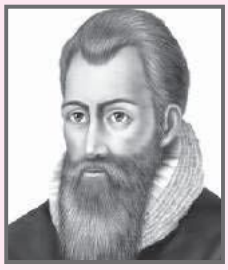
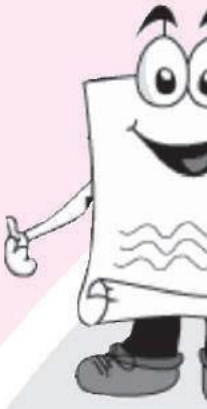
FÜN ARCADE

John Napier



John Napier was a very famous mathematician of his time and he was born in 1550 in Edinburgh, Scotland. His father was Sir Archibald Napier. Logarithms and the decimals' modern notations were introduced by him. He was very bright and he got admitted in the University of St. Andrews only when he was thirteen years old. It is also said that he had probably also studied at some universities in France and Italy etc.

For more than twenty years, Napier worked on a very complex that held a great value to Physical Science. A device named Napier's rods or bones shows creativeness of his mind in the field of Mathematics. Many mathematical functions like multiplication and division could be done mechanically. This device helped in analog computers and slide rules. Rabdologiae; seu Numerationes per Virgulas libri duo is the work published about his work in two volumes in 1617. He passed away the same year on 4th of April.



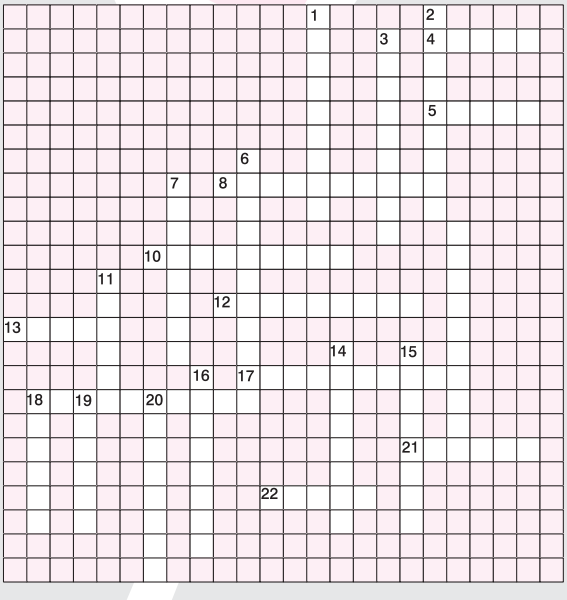
CROSSWORD

DOWN

1. A quadrilateral which has a pair of opposite sides which are parallel. (9)
2. The 20th and 80th percentiles of a set of data. (9)
3. The function being integrated in either a definite or indefinite integral. (9)
6. A selection of objects in which the order of the objects matters. (11)
7. A polynomial with one term. (8)
9. A quantity that can change or that may take on different values. (8)
11. The region between two concentric circles which have different radii. (6)
14. The determinant obtained by deleting the row and column of a given element of a matrix or determinant. (8)
15. The radius of a plane figure's inscribed circle. (8)
16. The horizontal line drawn as part of a fraction or radical. (8)
18. A unit for measuring angles. (6)
19. The number 10^{100}. (6)
20. A wave shaped like the graph of $y = \sin x$. (8)

ACROSS

4. Combining the elements of two or more sets. (5)
5. A doughnut shape. (5)
8. A curve usually expressed in polar coordinates that resembles a figure eight. (10)
10. A statement accepted as true without proof. (9)
12. The product of a given integer and all smaller positive integers. (9)
13. A special point used to construct and define a conic section. (5)
17. At a 90° angle. (10)
18. The process of finding a regression equation. (10)
21. The set of values of the independent variable(s) for which a function or relation is defined. (6)
22. A statement accepted as true without proof. (5)



Puzzles to Puzzle You

1. A friend of mine in London has a very nice cellar. He has two large barrels in the cellar. The larger barrel is mostly empty. But the smaller barrel is only 5/6th full of wine while it can hold 536 L. Supposing he empties the smaller barrel and fills the bigger barrel to find that the wine fills only 4/9th of it, how much wine would the larger barrel hold when full?

2. Suppose you had five billion rupees and you gave away a ₹ 500 note every minute, how long would it take you to give away all your money?

3. The sum of two numbers is ten. Their product is twenty. Can you find the sum of the reciprocals of the two numbers?

4. I found a very nice handbag in the shop that I thought I should simply must have. The price was ₹ 10. But I did not have the ₹ 10.
The storekeeper was known to me. He said that he had a credit system by which I could pay ₹ 1 only at the time of purchase and the balance ₹ 10 could be paid by me at the rate of ₹ 1 per week for 10 weeks.
Can you tell what annual rate of interest the storekeeper was charging me?

5. In the Sareen family, each daughter has the same number of brothers as she has sisters and each son has twice as many sisters as he has brothers. Now can you tell me how many sons and daughters do the Sareen family have?

6. I have a box. The two dimensions of the box are 4" and 3". Compute the third dimension of the box so that the space diagonal of the box is an integer.

SUDOKU

5				9		4		
			6			5		
				3	8		1	
			2				3	
2		4						
	1		8		3		9	
		2	4		5			
4		8			1			
		1		8		3		

December 2016 issue Answer

Crossword

						3 S													
						I													
			1 A			M				5 M		13 T	R	I	6 P	L	E		
			L			P				A					O				
			T			L				J					S				
11 C	10 O	N	I	C	2 S	E	4 C	T	I	O	N				I				
	R		T		Y	E	A			R	12 F	U	N	C	T	I	O	9 N	
	T		U		M	V	R			A					I			O	
	H		D		M	E	T			X					V			N	
	O		E		E	N	E			I					E			C	
	C				T	T	S			S								O	
	E				R		14 I											L	
	N				I		A	R	I	T	H	M	7 E	T	I	C		L	
	T				C		N						R					I	
	R				A		P					15 C	O	N	S	T	A	N	T
	E				L		L											E	
							A											A	
							N									16 Z	E	R	O
		8 F	R	E	Q	U	E	N	C	Y									
		O																	
		U																	
		7 R	E	C	I	P	R	O	C	A	L								

SUDOKU

8	5	4	2	1	9	7	6	3
3	9	7	8	6	5	4	2	1
2	6	1	4	7	3	9	8	5
7	8	5	1	2	6	3	9	4
6	4	9	5	3	8	1	7	2
1	3	2	9	4	7	8	5	6
9	2	6	3	8	4	5	1	7
5	1	3	7	9	2	6	4	8
4	7	8	6	5	1	2	3	9

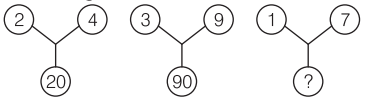
Puzzles to Puzzle you

1. 7 yr, 10 yr, 16 yr
3. 6 trains
5. 8.25 am
6. Older man
7. Same weight

HURRY! WIN ₹1000 IN CASH

JUST SOLVE & SEND KNOWLEDGE COEFFICIENT QUIZZER (NO. 26)

1. A staircase has a total of 5 steps. If a person covers a minimum of one and maximum of 3 steps in each stride. In how many ways can he climb the staircase?
2. ΔABC is right angled at A. The circle with centre A and radius AB cuts BC and AC internally at D and E, respectively. If $BD = 20$ cm and $CD = 16$ cm, then find the length of AC.
3. There are two clocks on a wall, both set to show the correct time at 6 am. One clock loss one minute in an hour while the other gains one minute in an hour. By how many minutes do the two clocks differ at 8 pm?
4. There are red balls and blue balls in an urn. The number of red balls is either 7, 9, 11, 13 or 15. If two balls are randomly choosed from the urn, then the probability they are different colour, is 1/2. How many red balls are in the urn?
5. Find the missing number.



6. Find the next term of the sequence
   46080, 3840, 384, 48, 8, 2, ?
7. Find the sum of the series
   $(x + y) + (x^2 + xy + y^2)$
   $+ (x^3 + x^2y + xy^2 + y^3) + \ldots$ to n terms.
8. Find the remainder when 110! is divided by 107^2.
9. A supreme court has given 6 to 3 decisions upholding a lower court. Find the number of ways it can give a majority decision reversing the lower court.
10.

A	1							9						...
B		2						8		10			...	...
C			3				7				11			...
D				4		6						12		14
E					5								13	

In which row 1790 will occur, if above is in a pattern?

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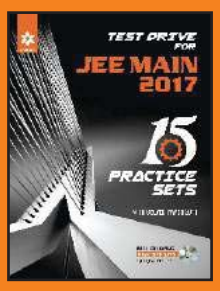

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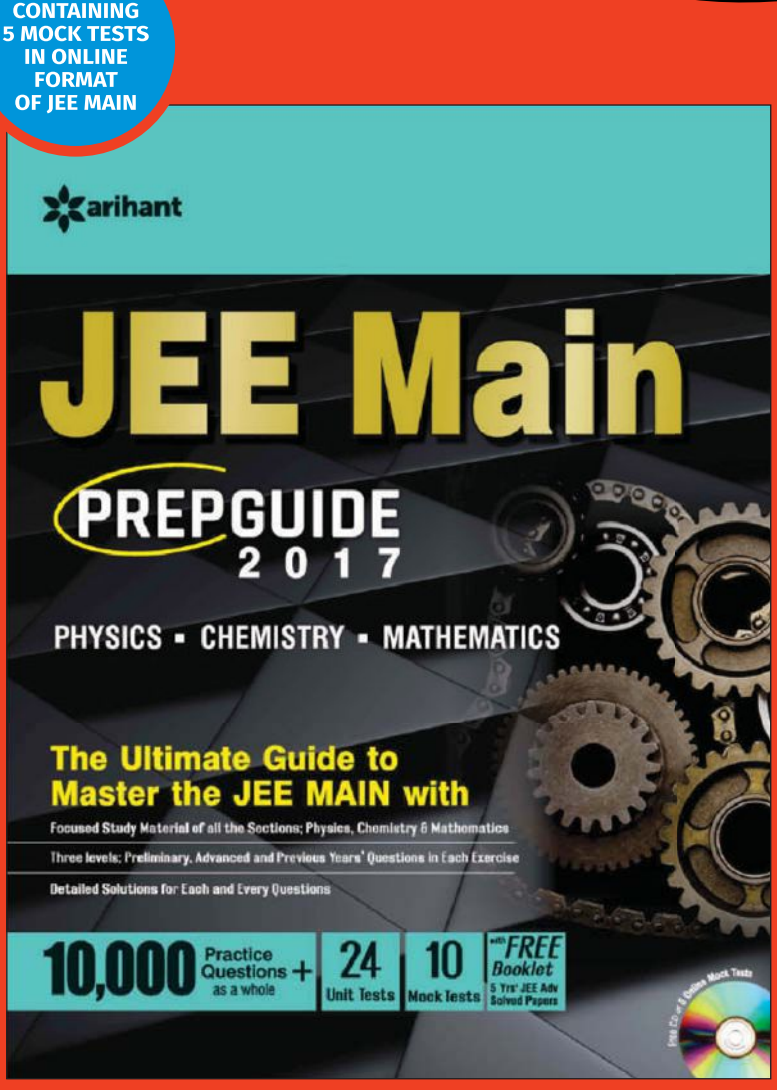
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